KOÇ UNIVERSITY

MATH 106 - CALCULUS I

FINAL January 3, 2014

Duration of Exam: 120 minutes

INSTRUCTIONS: CALCULATORS ARE NOT ALLOWED FOR THIS EXAM. No books, no notes, no questions and no talking allowed. You must always explain your answers and show your work to receive full credit. Use the back of these pages if necessary. Print (use CAPITAL LETTERS) and sign your name, and indicate your section below.

Surname, Name: ANSWER KEY

Signature: ______________________

Section (Check One):

Section 1: E. Ş. Yazıcı (Mon-Wed 16:00) —
Section 2: E. Ş. Yazıcı (Mon-Wed 13:00) —
Section 3: Doğan Bilge (Mon-Wed 11:30) —
Section 4: Doğan Bilge (Mon-Wed 14:30) —
Section 5: Altan Erdoğan (Tu-Th 16:00) —

PROBLEM	POINTS	SCORE
1	24	
2	20	
3	15	
4	15	
5	15	
6	15	
TOTAL	104	

1. Determine whether the following series are convergent or not.

a) (8 points) $\sum_{n=1}^{\infty}\left(\frac{\ln n}{\sqrt{n}}\right)$

$$\frac{\ln n}{\sqrt{n}} \geqslant \frac{1}{\sqrt{n}}$$

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent by p-series test, $(p=\frac{1}{2} \leq 1)$

$\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$ is divergent by comparison test.

b) (8 points) $\sum_{n=1}^{\infty}\left(\frac{n^2-1}{n^2}\right)^n$

$$\lim_{n\to\infty}\left(\frac{n^2-1}{n^2}\right)^n = \lim_{n\to\infty}\left(n-\frac{1}{n}\right)^n\left(n+\frac{1}{n}\right)^n$$

By definition, $\lim_{n\to\infty}\left(n-\frac{1}{n}\right)^n = e^{-1}$ and $\lim_{n\to\infty}\left(n+\frac{1}{n}\right)^n = e^1$

So, $\lim_{n\to\infty}\left(\frac{n^2-1}{n^2}\right)^n = 1$. Thus, $\sum_{n=1}^{\infty}\left(\frac{n^2-1}{n^2}\right)^n$ is divergent by divergency test

c) (8 points) $\sum_{n=0}^{\infty}\left(\frac{n\sin n}{n^4+1}\right)$

Sin n takes both negative and positive values. So, absolute convergence should be investigated.

$$\left|\frac{n \cdot \sin n}{n^4+1}\right| \leqslant \frac{n}{n^4+1} \leq \frac{n}{n^4} = \frac{1}{n^3} \qquad \sum_{n=1}^{\infty}\frac{1}{n^3}$$

$\downarrow$ convergent by p-series test $(p=3>1)$

So, $\sum_{n=0}^{\infty}\left|\frac{n \cdot \sin}{n^4+1}\right|$ is convergent by comparison test.

Since the series is absolutely convergent, it is convergent.

2. a) (8 points) Find the Maclaurin series of $f(x)$ if $f(0) = 0$ and $f'(x) = \dfrac{e^x - 1}{x}$.

$$f(x) = \int_0^x \frac{e^t - 1}{t}\,dt \qquad e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots$$

$$= \int_0^x \left(1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \cdots\right)dt \quad \text{(by McLaurin series expansion of } e^x\text{)}$$

$$= x + \frac{x^2}{2!\cdot 2} + \frac{x^3}{3!\cdot 3} + \frac{x^4}{4!\cdot 4} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n!\cdot n}$$

b) (4 points) Use the 4th degree Maclaurin polynomial of $f(x)$ to estimate $f(0.1)$.

$$f(x) \cong x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96}$$

$$f(0.1) \cong 0.1 + \frac{0.1^2}{4} + \frac{0.1^3}{18} + \frac{0.1^4}{96} \cong 0.103$$

c) (8 points) Find all values of x for which the power series $\sum_{n=1}^{\infty} \left(\dfrac{(x+2)^n}{n4^n}\right)$ is absolutely convergent, conditionally convergent or divergent. What is the radius of convergence of this series?

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x+2|^{n+1}}{(n+1)\cdot 4^{n+1}} \cdot \frac{n\cdot 4^n}{|x+2|^n} = \lim_{n\to\infty} \frac{|x+2|}{4}\cdot\frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{|x+2|}{4} \cdot \underbrace{\lim_{n\to\infty} \frac{n}{n+1}}_{1} = \lim_{n\to\infty} \frac{|x+2|}{4}$$

$$\frac{|x+2|}{4} < 1 \Rightarrow |x+2| < 4 \Rightarrow -4 < x+2 < 4 \Rightarrow -6 < x < 2$$

Result: The series is abs. convergent for $-6 < x < 2$, conditionally convergent for $x = -6$ and radius of convergence is 4.

For $x = 2$, $\sum_{n=1}^{\infty} \frac{4^n}{n\cdot 4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

For $x = -6$, $\sum_{n=1}^{\infty} \frac{(-4)^n}{n\cdot 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

3. a-)(10 points) Let $\{x_n\}$ be a sequence that satisfies the following:

- Each term is a *natural number*, i.e. $x_n \in \{0, 1, 2, 3, ...\}$ for each n;
- $x_n \neq x_m$ for $m \neq n$, i.e. each term of the sequence is a different natural number.

Show that $\{x_n\}$ is divergent.

Assume that the sequence converges to L.

$$\lim_{n\to\infty} x_n = L$$

Then, by definition, $\forall \varepsilon > 0, \exists N > 0$ s.t.

$|x_n - L| < \varepsilon$ for $n > N$.

Let $\varepsilon = \frac{1}{2}$. Then since $|x_n - L|$ has different positive integer values for $n > N$, at most one value of n satisfies the inequality which is a contradiction to our assumption.

b-)(5 points) If $\{x_n\}$ is a sequence where each term is a natural number, find a necessary and sufficient condition which will guarantee the convergence of $\{x_n\}$.

The sequence should ultimately stabilize. In other words, the sequence must have a constant value after a finite number of terms.

ultimately constant

4. (15 points) Find the shortest and the longest distance of the curve $\frac{x^2}{4} + y^2 = 1$ to the point $(1,0)$.

(Note: The distance between the points (a_1, a_2) and (b_1, b_2) is defined as $d = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$).

Let (a,b) be the extremum point. Then

$$d^2 = (a-1)^2 + b^2$$

Since $\frac{a^2}{4} + b^2 = 1$, $b^2 = 1 - \frac{a^2}{4}$

$$d^2 = (a-1)^2 + 1 - \frac{a^2}{4} = \frac{3a^2}{4} - 2a + 2$$

$d = \sqrt{\frac{3a^2}{4} - 2a + 2}$ for $-2 \leq a \leq 2$ since (a,b) is on the ellipse.

$$\frac{d(d)}{da} = \frac{\frac{3a}{2} - 2}{2\sqrt{\frac{3a^2}{4} - 2a + 2}} = 0 \Rightarrow a = \frac{4}{3}$$

Sign table:

		$4/3$	
f':	$-$	0	$+$
f:	$\searrow$		$\nearrow$
		minimum	

For minimum distance $a = \frac{4}{3}$

$$d = \sqrt{\frac{3a^2}{4} - 2a + 2} = \frac{\sqrt{6}}{3}$$

For longest distance, the boundary values should be applied. $a = -2 \Rightarrow d = 3$ So, $d = 3$ is the maximum distance

$a = 2 \Rightarrow d = 1$

The shortest and the longest distance are $\frac{\sqrt{6}}{3}$ and 3 respectively

5. (15 points) Show that $x^5 + 2x^3 + 2 + e^x = 0$ has *exactly* one root.

Let $x=-1 \Rightarrow (-1)^5 + 2\cdot(-1)^3 + 2 + e^{-1} = -1 + \frac{1}{e} < 0$

Let $x = 1 \Rightarrow 1^5 + 2\cdot 1^3 + 2 + e^1 = e + 5 > 0$

Since $x^5 + 2x^3 + 2 + e^x$ is continuous, by I.V.T.,

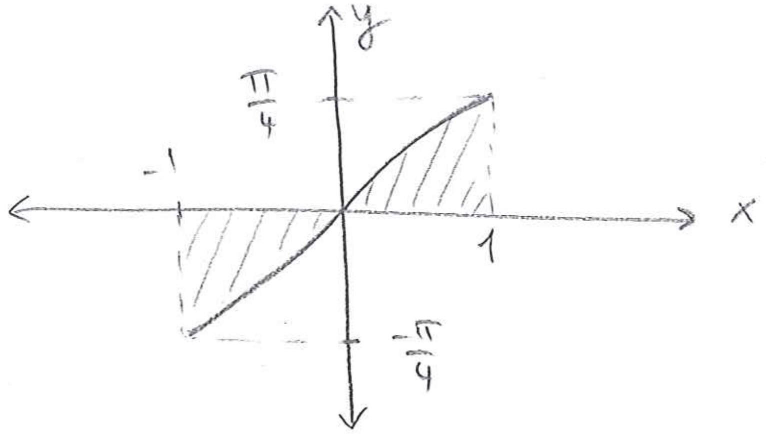
it has a root in $(-1, 1)$.

$f'(x) = 5x^4 + 6x^2 + e^x > 0$ for every values of x.

So, f is an increasing function. Thus, it is one-to-one

Since it has at least one root as proven above, it has <u>exactly</u> one root which takes place in the interval $(-1, 1)$.

6. (15 points) Find the area of the region enclosed by the curve $y = x^2 \tan^{-1} x$, the x-axis and the lines $x = -1$ and $x = 1$. (Note: $tan^{-1} = \arctan x$)



Since the function is an odd function, the graph is symmetric w.r.t. y-axis. The total area is found by doubling the area found by taking the integral from x=0 to x=1.

$$A = 2 \cdot \int_0^1 x^2 \tan^{-1} x \, dx$$

Let $u = \tan^{-1} x$ $\qquad dv = x^2 dx$

$du = \frac{1}{1+x^2} dx \qquad v = \frac{x^3}{3}$

$$\int_0^1 x^2 \tan^{-1} x \, dx = \tan^{-1} x \cdot \frac{x^3}{3} \Big|_0^1 - \frac{1}{3} \int_0^1 \frac{x^3}{1+x^2} dx$$

$$= \frac{\pi}{12} - \frac{1}{3} \cdot \int_0^1 x - \frac{x}{1+x^2} dx$$

$$= \frac{\pi}{12} - \frac{1}{3} \cdot \left(\frac{x^2}{2} - \frac{\ln |1+x^2|}{2} \right) \Big|_0^1$$

$$= \frac{\pi}{12} - \frac{1}{3} \cdot \left(\frac{1}{2} - \frac{\ln 2}{2} \right)$$

$$= \frac{\pi}{12} - \frac{1}{6} + \frac{\ln 2}{6} = \frac{\pi - 2 + \ln 4}{12} \Rightarrow A = 2 \cdot \frac{\pi - 2 + \ln 4}{12} = \frac{\pi - 2 + \ln 4}{6}$$