KOÇ ÜNİVERSİTESİ

MATH 503 APPLIED MATHEMATICS

Answers to HW Set #2

9.1.10) $\vec{F} = \cosh x\,\vec{i} + \sinh y\,\vec{j} + e^{z}\,\vec{k}$

$C: \vec{r}(t) = t\,\vec{i} + t^2\vec{j} + t^3\vec{k}$

from $(0,0,0)$ to $(2,4,8)$
$t=0$ $t=2$

$\vec{r}'(t) = \vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k}$

$\vec{F}\cdot\vec{r}'(t) = \cosh x(t) + 2t\sinh y(t) + 3t^2 e^{z(t)}$

$= \cosh t + 2t\sinh t^2 + 3t^2 e^{t^3}$

$\therefore$ Work done $= \int_0^2 \left(\cosh t + 2t\sinh t^2 + 3t^2 e^{t^3}\right) dt$

$= \int_0^2 \cosh t\, dt + \int_0^4 \sinh t^2\, dt^2 + \int_0^8 e^{t^3} dt^3$

$= \sinh t\Big|_0^2 + \cosh t^2\Big|_0^4 + e^{t^3}\Big|_0^8$

$= \sinh(2) + \cosh(4) - 1 + e^8 - 1$

9.1.15) $f = x^2 + y^2 + z^2$

$C: \vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + 2t\,\vec{k}$ $\quad 0 \leq t \leq 4\pi$

$\vec{r}'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + 2\vec{k}$

$\therefore s(t) = \int_0^t \sqrt{\sin^2 t + \cos^2 t + 4}\, dt = \sqrt{5}\, t$

Then $x = \cos t = \cos \frac{s}{\sqrt{5}}$

$y = \sin t = \sin \frac{s}{\sqrt{5}}$

$z = \frac{2}{\sqrt{5}} s$

$\int f(\vec{r})\, ds = \int_0^{4\pi\sqrt{5}} \left(\cos^2\frac{s}{\sqrt{5}} + \sin^2\frac{s}{\sqrt{5}} + \frac{4}{5}s^2\right) ds = 4\pi\sqrt{5}\left(1 + \frac{64\pi^2}{3}\right).$

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9.2.3) $\int_{(-1,5)}^{(4,3)} (3z^2 dx + 6xz\, dz) = \int_{(-1,5)}^{(4,3)} df = f(4,3) - f(-1,5)$
$= 183.$

where $f(x,z) = 3xz^2$

$f(4,3) = 3\cdot 4\cdot 9 = 108$

$f(-1,5) = 3(-1)\cdot 25 = -75$

Check:

$\vec{\nabla} f = 3z^2\,\vec{i} + 6xz\,\vec{k}$

$\therefore df = \vec{\nabla} f\cdot d\vec{r} = 3z^2 dx + 6xz\, dz.$

9.2.7) $\int_{(0,-1,1)}^{(2,4,0)} e^{x-y+z^2}(dx - dy + 2z\,dz) = \int_{(0,-1,1)}^{(2,4,0)} df$

$= f(2,4,0) - f(0,-1,1)$

where

$f(x,y,z) = e^{x-y+z^2}$ $\quad = -2\sinh(2)$

Since $f(2,4,0) = e^{-2}$, $f(0,-1,1) = e^{2}$

9.2.11) Given $2xy^2 dx + 2x^2y\, dy + dz$

Let $f(x,y,z) = x^2y^2 + z$

$\Rightarrow df = \vec{\nabla} f\cdot d\vec{r} = 2xy^2 dx + 2x^2y\, dy + dz$

$\therefore$ path independent

Other way:

Read $\vec{F} = 2xy^2\,\vec{i} + 2x^2y\,\vec{j} + \vec{k}$

$\Rightarrow \vec{\nabla}\times\vec{F} = 0$ $\quad\therefore$ path independent

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9.4.6 $\vec F = \sin y\,\vec i + \cos x\,\vec j$

R: triangle with vertices at

$A:(0,0)$, $B:(\pi,0)$, $C:(\pi,1)$

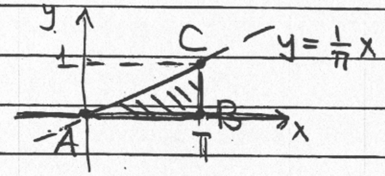
$\vec r_A = \vec 0$, $\vec r_B = \pi\vec i$, $\vec r_C = \pi\vec i + \vec j$

$\vec r_B - \vec r_A = \pi\vec i$, $\vec r_C - \vec r_A = \pi\vec i + \vec j$

$\vec N = (\vec r_B - \vec r_A)\times(\vec r_C - \vec r_A) = \pi\vec k$

$\therefore$ Unit normal vector to R: $\vec n = \vec k$

Area of the triangle R $= \frac{1}{2}|\vec N| = \pi/2$



$F_1 = \sin y$, $F_2 = \cos x$

$\Rightarrow -\frac{\partial F_1}{\partial y} = -\cos y$, $\frac{\partial F_2}{\partial x} = -\sin x$

$$\therefore \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = -\int_{x=0}^{\pi}\left(\int_{y=0}^{\frac{1}{\pi}x}(\sin x + \cos y)\,dy\right)dx$$

$$= -\int_0^{\pi}\left(\sin x\cdot\frac{x}{\pi} + \sin\frac{x}{\pi}\right)dx$$

$$= -\frac{1}{\pi}\int_0^{\pi} x\sin x\,dx - \pi\int_0^1 \sin\frac{x}{\pi}\,d\left(\frac{x}{\pi}\right)$$

$$= \pi\cos\frac{1}{\pi} - \pi - 1$$

Note:

$$\oint_{\partial R}\vec F\cdot d\vec r = \int_{\overline{AB}}(\vec F\cdot d\vec r)\Big|_{\overline{AB}} + \int_{\overline{BC}}(\vec F\cdot d\vec r)\Big|_{\overline{BC}} + \int_{\overline{CA}}(\vec F\cdot d\vec r)\Big|_{\overline{CA}}$$

Go around in counter-clockwise sense

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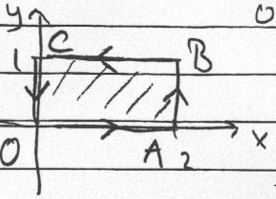
9.4.14 $W = e^x + e^y$ R: rectangle $0\le x\le 2$, $0\le y\le 1$

$\vec\nabla W = e^x\,\vec i + e^y\,\vec j$

$\vec n = \frac{dy}{ds}\vec i - \frac{dx}{ds}\vec j$



$$\therefore \frac{\partial W}{\partial s} = \vec n\cdot\vec\nabla W = e^x\frac{dy}{ds} - e^y\frac{dx}{ds}$$

$$\nabla^2 W \equiv \vec\nabla\cdot\vec\nabla W = e^x + e^y$$

1) $$\iint_R (e^x + e^y)\,dx\,dy = \int_{x=0}^{2}\int_{y=0}^{1}(e^x + e^y)\,dx\,dy$$

$$= (e^2 - 1) + 2(e-1) = e^2 + 2e - 3$$

2) $$\oint_C \frac{\partial W}{\partial s}ds = \left(\int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}\right)\left(e^x\frac{dy}{ds} - e^y\frac{dx}{ds}\right)ds$$

$$= \int_0^2(-1)\,dx + \int_0^1(e^2)\,dy + \int_2^0(e)(-dx) + \int_1^0(1)(-dy)$$

$$= -\int_0^2 dx + e^2\int_0^1 dy + e\int_0^2 dx + \int_0^1 dy$$

$$= e^2 + 2e - 3.$$

9.5.4 Paraboloid of revolution

$\vec r(u,v) = u\cos v\,\vec i + u\sin v\,\vec j + u^2\vec k$

$\frac{\partial\vec r}{\partial u} = \cos v\,\vec i + \sin v\,\vec j + 2u\,\vec k$ i) u=const. curve

$\vec r(v) = u_0(\cos v\,\vec i + \sin v\,\vec j) + \dots$

$\frac{\partial\vec r}{\partial v} = -u\sin v\,\vec i + u\cos v\,\vec j$

Circle of radius u_0 at $z = u_0^2$

$\vec N = \frac{\partial\vec r}{\partial u}\times\frac{\partial\vec r}{\partial v} = u\cos^2 v\,\vec k$ ii) V=const. curves:

$\vec r(u) = \cos v_0\,u\,\vec i + \sin v_0\,u\,\vec j + u^2\vec k$

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9.5.10) Helicoid. $\vec{r}(u,v) = u\cos v\,\vec{i} + u\sin v\,\vec{j} + v\vec{k}$

$u=$const. curve is a right-circular helix of radius u_0

$\frac{\partial \vec{r}}{\partial u} = \cos v\,\vec{i} + \sin v\,\vec{j}$, $\frac{\partial \vec{r}}{\partial v} = -u\sin v\,\vec{i} + u\cos v\,\vec{j} + \vec{k}$

$\Rightarrow \vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \sin v\,\vec{i} - \cos v\,\vec{j} + u\vec{k}$, $|\vec{N}| = \sqrt{1+u^2}$

$\therefore \vec{n} = \frac{\vec{N}}{|\vec{N}|} = \frac{\sin v}{\sqrt{1+u^2}}\vec{i} - \frac{\cos v}{\sqrt{1+u^2}}\vec{j} + \frac{u}{\sqrt{1+u^2}}\vec{k}$

9.5.28) $g(x,y,z) = x^2 - y^2 + z^2 - 1 = 0$

$\vec{\nabla} g = 2x\vec{i} - 2y\vec{j} + 2z\vec{k}$, $|\vec{\nabla} g| = 2\sqrt{x^2+y^2+z^2} = 2r$

$\therefore \vec{n} = \frac{\vec{\nabla} g}{|\vec{\nabla} g|} = \frac{x}{r}\vec{i} - \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}$

9.6.3) $\vec{F} = e^{2y}\vec{i} + e^{-2z}\vec{j} + e^{2x}\vec{k}$

S: $\vec{r}(u,v) = 3\cos u\,\vec{i} + 3\sin u\,\vec{j} + v\vec{k}$

$0 \le u \le \pi/2$, $0 \le v \le 2$

$\frac{\partial \vec{r}}{\partial u} = -3\sin u\,\vec{i} + 3\cos u\,\vec{j}$, $\frac{\partial \vec{r}}{\partial v} = \vec{k}$

$\therefore \vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = 3\sin u\,\vec{j} + 3\cos u\,\vec{i}$

$$\iint_S (\vec{F}\cdot\vec{n})\,dA = \int_{v=0}^{2}\int_{u=0}^{\pi/2} \left(e^{6\sin u}\,3\cos u + e^{-3v}\,3\sin u + e^{\cancel{6\cos u}}\cdot 0\right) du\,dv$$

$= 2\sinh(6)$

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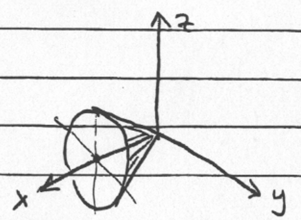
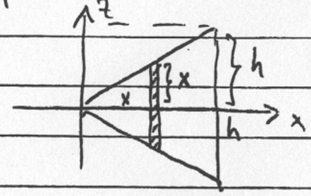
9.2.10) Cone. T : $y^2+z^2 \le x^2$, $0 \le x \le h$



Evaluate moment of inertia

$$I_x = \iiint_T (y^2+z^2)\,dx\,dy\,dz$$



At x , $\sqrt{y^2+z^2} = x$

Let $y = r\cos\theta$ $\quad 0 \le \theta \le 2\pi$

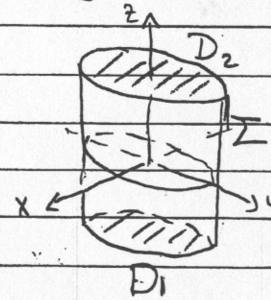
$z = r\sin\theta$ $\quad 0 \le r \le x$

$$\therefore I_x = \int_{x=0}^{h}\left(\int_{r=0}^{x}\int_{\theta=0}^{2\pi} r^2\cdot r\,dr\,d\theta\right)dx$$

$$= 2\pi\int_{x=0}^{h}\left(\frac{r^4}{4}\Big|_0^x\right)dx = \frac{\pi}{2}\int_0^h x^4\,dx = \frac{\pi}{10}h^5.$$

9.7.15) $\vec{F} = \cos y\,\vec{i} + \sin x\,\vec{j} + \cos z\,\vec{k}$

S : boundary of the region $x^2+y^2 \le 4$, $|z| \le 2$ (solid cylinder)



$S = D_1 \cup D_2 \cup \Sigma$

where

$D_1 = \{(x,y,z) \mid z=-2,\ x^2+y^2 \le 4\}$

with outward normal $\vec{n} = -\vec{k}$

$D_2 = \{(x,y,z) \mid z=2,\ x^2+y^2 \le 4\}$

with outward normal $\vec{n} = \vec{k}$

$\Sigma = \{(x,y,z) \mid x^2+y^2=4,\ -2 \le z \le 2\}$

outward normal

$\vec{n} = \frac{x}{2}\vec{i} + \frac{y}{2}\vec{j}$

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(cont.)

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = -\sin z$$

1) $$\iiint_T (\vec{\nabla}\cdot\vec{F})\, dV = \iiint (-\sin z)\, dz\, dx\, dy$$

$$= -\int_{-2}^{2} \sin z\, dz \left(\iint dx\, dy\right)$$

$$= \cos z \Big|_{-2}^{2} \left(\iint dx\, dy\right) = 0$$

2) by divergence theorem

$$\oiint_S (\vec{F}\cdot\vec{n})\, dA = \left(\iint_{D_1} + \iint_{D_2} + \iint_{\Sigma}\right)(\vec{F}\cdot\vec{n})\, dA$$

$$= -\iint_{D_1} \cos z\, dA + \iint_{D_2} \cos z\, dA + \iint_{\Sigma}\left(\frac{x\cos y}{2} + \frac{y \sin x}{2}\right) dA$$

$$= \cancel{-\cos z\, 4\pi} + \cancel{\cos z\, 4\pi} + \frac{1}{2}\iint_{\Sigma} (x\cos y + y \sin x)\, dA$$

On Σ: $x = 2\cos\theta$
$y = 2\sin\theta$ and $dA = 2\, d\theta\, dz$
$0 \leq \theta \leq 2\pi$

$$\therefore \iint_{\Sigma} (x\cos y + y \sin x)\, dA$$

$$= 16\int_{\theta=0}^{2\pi} \cos\theta \cos(2\sin\theta)\, d\theta + 16\int_{\theta=0}^{2\pi} \sin\theta \sin(2\cos\theta)\, d\theta$$

$$= 8\int_{\theta=0}^{2\pi} \cos(2\sin\theta)\cdot d(2\sin\theta) - 8\int_{\theta=0}^{2\pi} \sin(2\cos\theta)\, d(2\cos\theta)$$

$$= 8 \sin(2\sin\theta)\Big|_{\theta=0}^{2\pi} + 8\cos(2\cos\theta)\Big|_{\theta=0}^{2\pi} = 0$$

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9.8.5) $\vec{F} = 9x\,\vec{i} + y\cosh^2 x\,\vec{j} - z\sinh^2 x\,\vec{k}$

$S: g(x,y,z) = 4x^2 + y^2 + 9z^2 - 36 = 0$

Boundary of an ellipsoid $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{6}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$

$$\vec{\nabla}g = 8x\,\vec{i} + 2y\,\vec{j} + 18z\,\vec{k}$$

$$|\vec{\nabla}g| = \sqrt{64x^2 + 4y^2 + 324z^2} \quad \therefore \vec{n} = \frac{\vec{\nabla}g}{|\vec{\nabla}g|}$$

$$\vec{\nabla}\cdot\vec{F} = 9 + \cosh^2 x - \sinh^2 x = 10$$

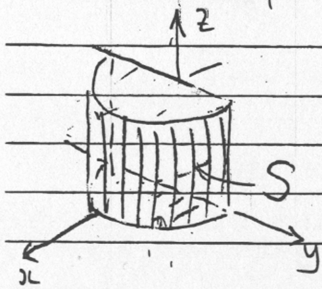
Evaluate $\iiint_T (\vec{\nabla}\cdot\vec{F})\, dV = \oiint_S (\vec{F}\cdot\vec{n})\, dA$

LHS $= 10 \iiint_T dV = 10\cdot$ (volume of the ellipsoid)

$$= 10 \times \frac{4}{3}\pi\; 3\times 6\times 2 = 480\pi$$

9.9.4) $\vec{F} = x\cos 2z\,\vec{k}$, $\vec{\nabla}\times\vec{F} = -\cos 2z\,\vec{j}$

$S = \{(x,y,z) \mid x^2+y^2=1,\ y \geq 0,\ 0 \leq z \leq \frac{\pi}{4}\}$



semi-cylinder with open ends

outward normal

$\vec{n} = x\vec{i} + y\vec{j}$, $x^2+y^2=1$

Evaluate $\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, dA = \oint_C \vec{F}\cdot d\vec{r}$

$= \iint_S (-\cos 2z)\frac{y}{\sqrt{x^2+y^2}}\, dA$ Result $= 0$.