9. Relations

Relations are discrete structures that are used to represent relationships between elements of sets.

Relations can be used to solve problems such as:

- Determining which pairs of cities are linked by airline flights in a network,
- Computing the distance between a pair of registered *Facebook* users.
- Finding an efficient order for different phases of a complicated project,
- Producing a useful way to store information in computer databases, etc.

9.1 Relations and Their Properties

Definition: **Binary relation**

Let A, B be sets. A binary relation R from A to B is a set of ordered pairs, hence a subset of $A \times B$.

Notation:

a is "related to" b by R:	$a\ R\ b$:	$(a,b) \in R$; $a \in A$, $b \in B$
a is "not related to" b by R:	$a \not R\ b$:	$(a,b) \notin R$

e.g.

A: set of cities
B: set of countries
R: $(a, b) \in R$ if city a is in country b.

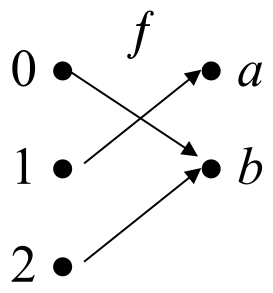
(Izmir, Turkey), (Paris, France) $\in R$

Function is a special case of relation

A function f from A to B can be thought of as the set of ordered pairs (a, b) s.t. $b = f(a)$

Since the function f is a subset of $A \times B$, f is a relation from A to B.

Function is a special case of relation: Every element of A is the first element of exactly one ordered pair of the function f.



Relations defined on a single set:

Definition:
A relation on a set A is a relation from A to A.

e.g.

$A = \{1, 2, 3, 4\}$

$R = \{(a, b) \mid a \mid b, (a, b) \in A \times A\}$

$= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$

Relations defined on a single set:

Definition:
A relation on a set A is a relation from A to A.

e.g.

$$A = \{1, 2, 3, 4\}$$
$$R = \{(a, b) \mid a \mid b, (a, b) \in A \times A\}$$
$$= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$$

e.g.
How many relations are there on a set with n elements?

$|A \times A| = n^2$

$\therefore 2^{n^2}$ (# of subsets of $A \times A$)

Properties of Relations defined on a set:

Definition:
A relation R on a set A is called **reflexive** iff

$$(a, a) \in R \quad \forall a \in A$$

e.g.

$A = \{1, 2, 3\}$

$R_1 = \{(1, 2), (2, 2), (1, 3)\}$ (not reflexive)
$R_2 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$ (reflexive)
$R_3 = \{(1, 3), (3, 1)\}$ (irreflexive)

Properties of Relations defined on a set:

Definition:
A relation R on a set A is called **reflexive** iff

$$(a, a) \in R \quad \forall a \in A$$

e.g.

$A = \{1, 2, 3\}$

$R_1 = \{(1, 2), (2, 2), (1, 3)\}$ (not reflexive)
$R_2 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$ (reflexive)
$R_3 = \{(1, 3), (3, 1)\}$ (irreflexive)

e.g.
R: The set of pairs of people having the same eye color (reflexive)

Definition:
A relation R on a set A is called

symmetric iff the following holds

$$(b,a)\in R \rightarrow (a,b)\in R \quad \forall a,b\in A$$

Definition:

A relation R on a set A is called

symmetric iff the following holds

$$(b,a)\in R \rightarrow (a,b)\in R \quad \forall a,b\in A$$

anti-symmetric iff the following holds

$$(a,b)\in R \text{ and } (b,a)\in R \rightarrow a = b \quad \forall a,b\in A$$

Definition:
A relation R on a set A is called

symmetric iff the following holds

$$(b,a)\in R \rightarrow (a,b)\in R \quad \forall a,b\in A$$

anti-symmetric iff the following holds

$$(a,b)\in R \text{ and } (b,a)\in R \rightarrow a = b \quad \forall a,b\in A$$

e.g.
$R_t = \{(a,b) \mid a \text{ is taller than } b\}$ anti-symmetric

$R = \{(a,b) \mid a+b+ab = 12;\ a,b \in Z\}$ symmetric

Definition:
A relation R on a set A is called

symmetric iff the following holds
$$(b,a)\in R \rightarrow (a,b)\in R \quad \forall a,b\in A$$

anti-symmetric iff the following holds
$$(a,b)\in R \text{ and } (b,a)\in R \rightarrow a = b \quad \forall a,b\in A$$

e.g.
$R_t = \{(a,b) \mid a \text{ is taller than } b\}$ anti-symmetric

$R = \{(a,b) \mid a+b+ab = 12;\ a,b \in Z\}$ symmetric

asymmetric iff $\forall a,b\in A \ (a,b)\in R \rightarrow (b,a)\notin R$

Definition:

R on set A is called **transitive** iff

$(a,b)\in R$ and $(b,c)\in R \rightarrow (a,c)\in R \quad \forall a,b,c \in A.$

e.g.

$R_t = \{(a,b) \mid a$ is taller than $b\}$ transitive?

Definition:

R on set A is called **transitive** iff

$$(a,b)\in R \text{ and } (b,c)\in R \rightarrow (a,c)\in R \quad \forall a,b,c \in A.$$

e.g.

$R_t = \{(a,b) \mid a \text{ is taller than } b\}$ transitive?

e.g.

$A = \{1, 2, 3\}$

$R_1 = \{(1, 2), (2, 3), (1, 3)\}$ (transitive)

$R_2 = \{(1, 2), (2, 3)\}$ (not transitive)

$R_3 = \{(1, 2)\}$ (?)

e.g.
How many reflexive relations are there on a set with n elements?

If R is reflexive, then:

there are n pairs such that $(a,a) \in R$

and $n(n-1)$ pairs such that $(a,b) \in R$ where $a \neq b$

$\Rightarrow$ # of reflexive relations $= 2^{n(n-1)}$

e.g.
How many symmetric relations are there on a set with n elements? (Exercise)

Combining relations:

Let $A = \{a, b\}$ $B = \{1, 2, 3\}$

$R_1 = \{(a, 1), (b, 3)\}$
$R_2 = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$
$R_3 = \{(b, 1), (b, 2)\}$
$R_4 = \{(a, 1), (b, 2)\}$

$R_1 \cup R_3 = \{(a, 1), (b, 1), (b, 2), (b, 3)\}$
$R_1 \cap R_2 = \{(a, 1)\}$
$R_2 - R_3 = \{(a, 1), (a, 2)\}$
$R_1 \oplus R_4 = \{(b, 2), (b, 3)\}$

$\oplus$ is called "symmetric difference", acts like XOR

Definition: Let R: $A \rightarrow B$ and S: $B \rightarrow C$. Then the **composite relation** of R and S,

$S \circ R$: $A \rightarrow C$ is defined s.t.

$(a, c) \in S \circ R$ iff $(a, b) \in R$ and $(b, c) \in S$.

Definition: Let $R: A \rightarrow B$ and $S: B \rightarrow C$. Then the **composite relation** of R and S,

$S \circ R: A \rightarrow C$ is defined s.t.

$(a, c) \in S \circ R$ iff $(a, b) \in R$ and $(b,c) \in S$.

Definition:
Let R be a relation on A.
The **powers** R^n, $n = 1, 2, 3, \ldots$, are defined by
$R^1 = R,\ R^2 = R \circ R, \ldots\ R^n = R^{n-1} \circ R$.

e.g. $R = \{(a,b) \mid b$ is a **parent** of $a\}$
$\Rightarrow R^2 = \{(a,c) \mid c$ is a **grand-parent** of $a\}$ why?
since $(a,b) \in R$ means "b is a parent of a", and $(b,c) \in R$ means "c is a parent of b".

Theorem:

R on a set A is transitive iff $R^n \subseteq R$ for all $n = 1, 2, 3, \ldots$

Theorem:
R on a set A is transitive iff $R^n \subseteq R$ for all $n = 1, 2, 3, \ldots$

Proof:

If part: (if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$, then R is transitive)
If $R^n \subseteq R$, in particular $R^2 \subseteq R$.
Then, if $(a,b) \in R$ and $(b,c) \in R$, by definition $(a,c) \in R^2$. Since $R^2 \subseteq R$, $(a,c) \in R$.
$\therefore$ R is transitive.

Theorem:

R on a set A is transitive iff $R^n \subseteq R$ for all $n = 1, 2, 3, \ldots$

Proof:

If part: (if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$, then R is transitive)

If $R^n \subseteq R$, in particular $R^2 \subseteq R$.

Then, if $(a,b) \in R$ and $(b,c) \in R$, by definition $(a,c) \in R^2$. Since $R^2 \subseteq R$, $(a,c) \in R$.

$\therefore$ R is transitive.

Only if part: (If R is transitive, then $\forall n \;\; R^n \subseteq R$) Use **induction** on n.

Theorem:

R on a set A is transitive iff $R^n \subseteq R$ for all $n = 1, 2, 3, \ldots$

Proof:

If part: (if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$, then R is transitive)
If $R^n \subseteq R$, in particular $R^2 \subseteq R$.
Then, if $(a,b) \in R$ and $(b,c) \in R$, by definition $(a,c) \in R^2$. Since $R^2 \subseteq R$, $(a,c) \in R$.
$\therefore$ R is transitive.

Only if part: (If R is transitive, then $\forall n \; R^n \subseteq R$) Use **induction** on n.
Basis step: $R^1 \subseteq R$; true for $n = 1$.
Inductive step: Assume $R^n \subseteq R$ and R is transitive. Show $R^{n+1} \subseteq R$.

Theorem:

R on a set A is transitive iff $R^n \subseteq R$ for all $n = 1, 2, 3,\ldots$

Proof:

If part: (if $R^n \subseteq R$ for $n = 1, 2, 3,\ldots$, then R is transitive)
If $R^n \subseteq R$, in particular $R^2 \subseteq R$.
Then, if $(a,b)\in R$ and $(b,c)\in R$, by definition $(a,c)\in R^2$. Since $R^2 \subseteq R$, $(a,c)\in R$.
$\therefore$ R is transitive.

Only if part: (If R is transitive, then $\forall n \; R^n \subseteq R$) Use **induction** on n.
Basis step: $R^1 \subseteq R$; true for $n = 1$.
Inductive step: Assume $R^n \subseteq R$ and R is transitive. Show $R^{n+1} \subseteq R$.

Let $(a,b)\in R^{n+1} = R^n \circ R$.
Then $\exists x\in A$ s.t. $(a, x)\in R$ and $(x,b)\in R^n$. Since $R^n \subseteq R$, $(x,b) \in R$.
Since R is transitive and $(a, x)\in R$, we have $(a,b)\in R$

$$\therefore R^{n+1} \subseteq R$$

Inverse and Complementary:

Inverse of R: $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

Complementary of R: $\bar{R} = \{(a, b) \mid (a, b) \notin R\}$

Inverse and Complementary:

Inverse of R: $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

Complementary of R: $\bar{R} = \{(a, b) \mid (a, b) \notin R\}$

e.g.

Let $R = \{(a, b) \mid a < b\}$ $R: A \rightarrow B$.

Inverse of R: $R^{-1} = \{(b, a) \mid a < b\}$
Complementary of R: $\bar{R} = \{(a, b) \mid a \geq b\}$

e.g.

R, S are reflexive relations on A.

a) $R \cup S$ is reflexive? Yes,

since $(x, x) \in R$
so does $R \cup S$

b) $R \cap S$ is reflexive? ✓

c) $R \oplus S$ is irreflexive? ✓

d) R-S is irreflexive? ✓

e) $S \circ R$ is reflexive? ✓

f) R^{-1} is reflexive?

g) Complementary of R is irreflexive?

e.g.
Suppose R is **irreflexive**. Is R^2 also irreflexive?
No. Counter-example: Let $a \neq b$ and $R = \{(a, b), (b, a)\}$

9.2 *n*-ary Relations and Their Applications

Definition:
Let $A_1, A_2, \ldots, A_n$ be sets.
An *n*-ary relation on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$.

The sets A_i : Domains of the relation
n: Degree of the relation

e.g.

$$R = \{(a, b, c) \mid a < b < c\}$$

Databases and Relations

The way we organize information in a database is important.
Operations such as **add/delete** record, **update** records, **search** for record, all have heavy computation.
∴ Various methods for representing databases exist.

One method in particular is relational data model.

A database consists of records of n-tuples, made up of domains (fields).
e.g. Airflight Company (Flight No, Departure, Destination, Date)

You will have an elective database course in 3rd or 4th year.

9.3 Representing Relations

Definition: A relation R can also be represented by a matrix $\mathbf{M}_R = [m_{ij}]$:

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

e.g. Let $A = \{1, 2\}$, $B = \{a, b, c\}$ and R: $A \rightarrow B$ such that
$R = \{(1, b), (2, a), (2, b), (2, c)\}$

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

9.3 Representing Relations

Definition: A relation R can also be represented by a matrix $\mathbf{M}_R = [m_{ij}]$:

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

e.g. Let $A = \{1, 2\}$, $B = \{a, b, c\}$ and R: $A \rightarrow B$ such that
$R = \{(1, b), (2, a), (2, b), (2, c)\}$

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

e.g.
Let R be a relation defined on $A = \{1, 2, 3\}$: $R = \{(1, 2), (2, 2), (1, 3)\}$

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that we get a square matrix whenever R: $A \rightarrow A$.

- Reflexive relation R s.t. $(a_i, a_i) \in R$

$\Rightarrow \forall i \;\; m_{ii} = 1$

i.e. , $\mathbf{M}_R = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}$ diagonal with all ones

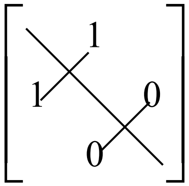
- Reflexive relation R s.t. $(a_i, a_i) \in R$

  $\Rightarrow \forall i \;\; m_{ii} = 1$

  i.e. , $$\mathbf{M}_R = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}$$ diagonal with all ones

- Symmetric relation R s.t. $(a_i, a_j) \in R \leftrightarrow (a_j, a_i) \in R$

  $\Rightarrow \forall i,j \;\; m_{ij} = m_{ji}$

  $\mathbf{M}_R =$ 

  Symmetric matrix
  ($\mathbf{M}_R = \mathbf{M}^{\mathrm{T}}_R$)

- Inverse and complementary relations:

If $\mathbf{M}_R = [m_{ij}]_{m\times n}$, then

<u>Inverse</u>: $\mathbf{M}_{R^{-1}} = [m_{ji}]_{n\times m}$ (transpose)

<u>Complementary</u>: $\mathbf{M}_{\overline{R}} = [\neg m_{ij}]_{m\times n}$ (negation)

Using Zero – One Matrices:

A matrix with entries that are either 0 or 1 is called a **zero-one matrix**.

Definition:

$\mathbf{A} = [a_{ij}]$ $\mathbf{B} = [b_{ij}]$ $m \times n$ zero-one matrices

Join of **A**, **B**: $\mathbf{A} \vee \mathbf{B} = [a_{ij} \vee b_{ij}]$
Meet of **A**, **B**: $\mathbf{A} \wedge \mathbf{B} = [a_{ij} \wedge b_{ij}]$

e.g.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Using Zero – One Matrices:

A matrix with entries that are either 0 or 1 is called a **zero-one matrix**.

Definition:

$\mathbf{A} = [a_{ij}]$ $\mathbf{B} = [b_{ij}]$ $m\times n$ zero-one matrices

Join of **A**, **B**: $\mathbf{A} \vee \mathbf{B} = [a_{ij} \vee b_{ij}]$
Meet of **A**, **B**: $\mathbf{A} \wedge \mathbf{B} = [a_{ij} \wedge b_{ij}]$

e.g.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Remark: Let $R_1: A \rightarrow B$ and $R_2: A \rightarrow B$

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2}$$

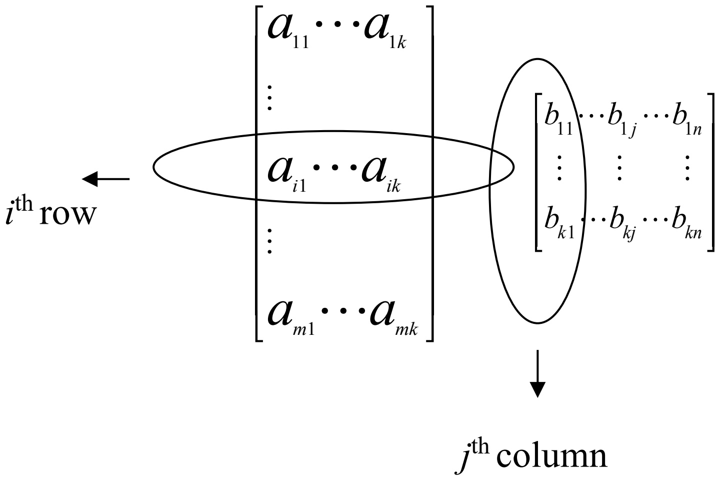
$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}$$

Definition: **Boolean product**

Let $\mathbf{A} = [a_{ij}] : m\times k$, $\mathbf{B} = [b_{ij}] : k\times n$ zero-one matrices

$$\mathbf{A} \odot \mathbf{B} = [c_{ij}] : m\times n, \text{ where}$$

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots. \vee (a_{ik} \wedge b_{kj})$$



e.g.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}_{3 \text{ x } 2} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2 \text{ x } 3}$$

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{3 \text{ x } 3}$$

e.g.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}_{3 \text{ x } 2} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2 \text{ x } 3}$$

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{3 \text{ x } 3}$$

Remark: Let R: $A \to B$ and S: $B \to C$

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S$$

Definition: ***r* th Boolean Power**

Let **A** be a square ($n \times n$) zero-one matrix and r be a positive integer.

$$\mathbf{A}^r = \underbrace{\mathbf{A} \odot \mathbf{A} \odot \ldots \odot \mathbf{A}}_{r \text{ times}}$$

$$\mathbf{A}^0 = \mathbf{I}_n$$

Remark: Let R: $A \rightarrow A$

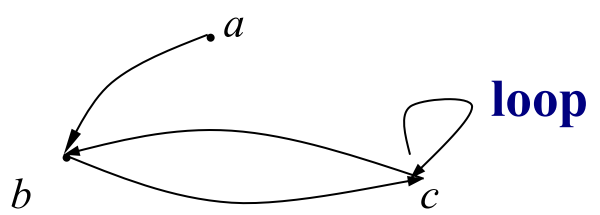
$$\mathbf{M}_{R^n} = [\mathbf{M}_R]^n$$

Representing Relations Using Graphs:

Pictorial representation.

Definition:
A **directed graph** (digraph) consists of a set V of **vertices** (or nodes) along with a set E of **edges** (or arcs) which are ordered pairs of vertices.

Edge(a, b): a is initial vertex (node), b is terminal vertex (node)
e.g.



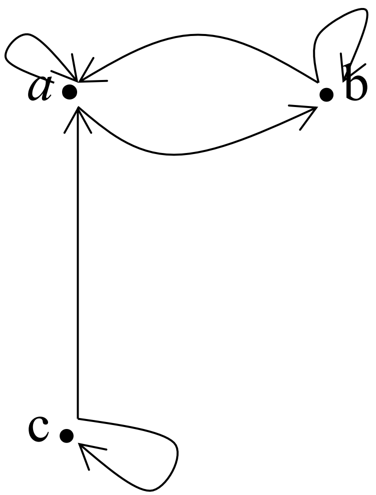
$R = \{(a, b), (b, c), (c, b), (c, c)\}$

Relation R on a set A is defined with
- i) elements of A: vertices (nodes)
- ii) ordered pairs $(a, b) \in R$: edges

Relation R is:

- reflexive iff every node has a loop
- symmetric iff every edge between two nodes has an edge in the opposite direction.
- transitive iff edge $(a, b) \wedge$ edge $(b, c) \rightarrow$ edge (a, c) $\forall a,b,c$

e.g.



reflexive

Example to graph representation of a relation:

Connectivity problems:

1) Which nodes are connected?
2) What is the shortest path between two nodes?

9.4 Closures of Relations

e.g. Let $R = \{(1,1), (1, 2), (3, 2)\}$ on $A = \{1, 2, 3\}$

R is not reflexive; what is the smallest possible reflexive relation containing R?

9.4 Closures of Relations

e.g. Let $R = \{(1,1), (1, 2), (3, 2)\}$ on $A = \{1, 2, 3\}$

R is not reflexive; what is the smallest possible reflexive relation containing R?

$S = \{(1, 1), (1, 2), (3, 2), \underline{(2, 2)}, \underline{(3, 3)}\}$

S is the reflexive **closure** of R.

Definition: **Closure**

Let R be a relation on A

P: some property, such as symmetry, reflexivity, transitivity

R may or may not have the property P.

The **closure** S is the smallest possible set with property P, which contains R.

Definition: **Closure**

Let R be a relation on A

P: some property, such as symmetry, reflexivity, transitivity

R may or may not have the property P.

The **closure** S is the smallest possible set with property P, which contains R.

More formal definition of **closure**:

If there is a relation S with property P containing R s.t. S is the subset of every relation with property P containing R, then S is called the **closure** of R with P.

Reflexive Closure:

Let $R = \{(1,1), (1, 2), (3, 2)\}$ on $A = \{1, 2, 3\}$

The smallest possible reflexive relation containing R:

$$S = \{(1, 1), (1, 2), (3, 2), \underline{(2, 2)}, \underline{(3, 3)}\}$$

S = Reflexive closure of $R = R \cup \Delta$,
where $\Delta = \{(a, a) \mid a \in A\}$: diagonal relation

Reflexive Closure:

Let $R = \{(1,1), (1, 2), (3, 2)\}$ on $A = \{1, 2, 3\}$

The smallest possible reflexive relation containing R:

$$S = \{(1, 1), (1, 2), (3, 2), \underline{(2, 2)}, \underline{(3, 3)}\}$$

S = Reflexive closure of $R = R \cup \Delta$,
where $\Delta = \{(a, a) \mid a \in A\}$: diagonal relation

e.g.

$R = \{(a, b) \mid a < b\}$, reflexive closure?

$$R \cup \Delta = \{(a, b) \mid a < b\} \cup \{(a, a) \mid a \in Z\}$$
$$= \{(a, b) \mid a \leq b\}$$

Symmetric Closure:

Let $R = \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 1), (3, 3)\}$ on $A = \{1, 2, 3\}$

We should add all ordered pairs (b,a) , where (a, b) is in R and (b, a) is not in R.
Symmetric closure of $R = R \cup \{(3, 2), (1, 3)\}$

Symmetric Closure:

Let $R = \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 1), (3, 3)\}$ on $A = \{1, 2, 3\}$

We should add all ordered pairs (b,a) , where (a, b) is in R and (b, a) is not in R.
Symmetric closure of $R = R \cup \{(3, 2), (1, 3)\}$

Symmetric closure of $R = R \cup R^{-1}$ (since $R^{-1} = \{(b, a) \mid (a, b) \in R\}$)

e.g.
$R = \{(a, b) \mid a < b\}$

Symmetric closure of $R = R \cup R^{-1}$

$$= \{(a, b) \mid a < b\} \cup \{(b, a) \mid a < b\}$$
$$= \{(a, b) \mid a \neq b\}$$

Transitive Closure:

Let $R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$ on $\{1, 2, 3, 4\}$
R is not transitive since there are pairs $(a, c) \notin R$ although $(a, b), (b, c) \in R$.

(i) $R \cup \{(1, 2), (2, 3), (2, 4), (3, 1)\}$
Is it transitive?

Transitive Closure:

Let $R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$ on $\{1, 2, 3, 4\}$

R is not transitive since there are pairs $(a, c) \notin R$ although $(a, b), (b, c) \in R$.

(i) $R \cup \{(1, 2), (2, 3), (2, 4), (3, 1)\}$

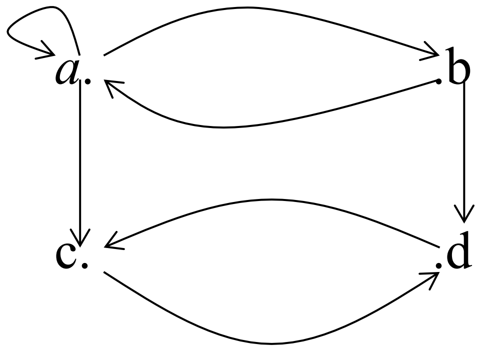
Is it transitive? NO!

It has (3, 1),(1, 4), but not (3, 4).

We have a more difficult problem!!!

We might repeat step (i) until reaching a transitive relation. But there are better ways.

e.g. Draw reflexive closure of



How about symmetric closure? Transitive closure?

Paths in Directed Graphs

We now introduce a new terminology that we will use in the construction of transitive closures.

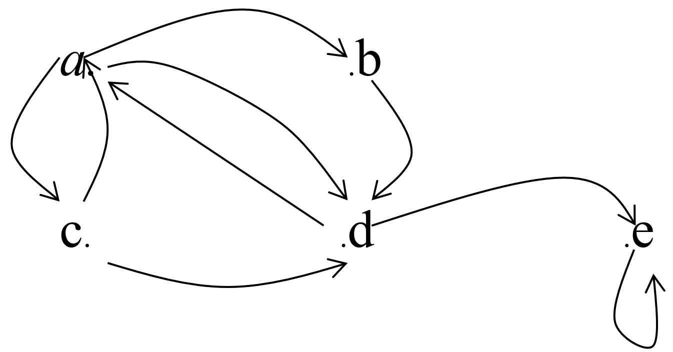
Definition:

A **path** from a to b in the directed graph G is a sequence of edges (x_0, x_1), (x_1, x_2), … (x_{n-1}, x_n) in G where $x_0 = a$ and $x_n = b$. This path is denoted by $x_0, x_1, \ldots, x_n$ and has a length of n.

If $x_0 = x_n$, the path is called a **cycle** or **circuit**.

Two vertices are said to be **connected** if there's a path between them.

e.g.



A path:

a, b, d, a, c

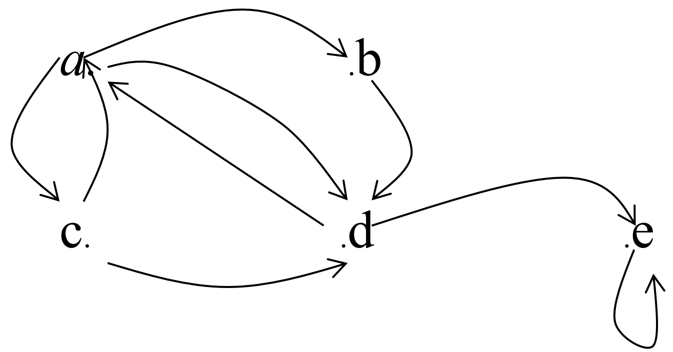
a is connected to e, but e is not connected to a.

The term **path** also applies to relations.

Theorem:

Let R be a relation on A, then there is a path of length n from a to b iff $(a, b)\in R^n$.

e.g.



A path:

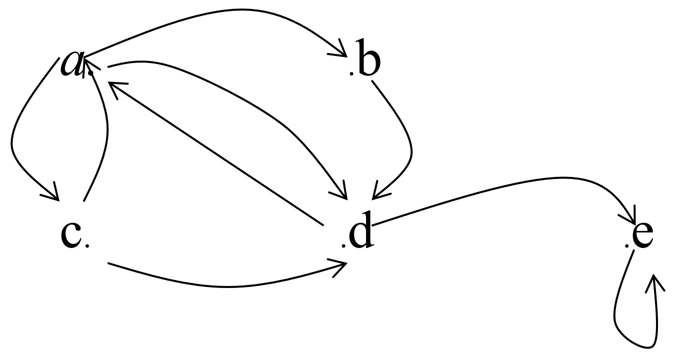
a, b, d, e

$(a,e)\in R^3$ since there is a path of length 3 between a and e.

Theorem:

Let R be a relation on A, then there is a path of length n from a to b iff $(a, b)\in R^n$.

e.g.



A path:
a, b, d, e

$(a,e)\in R^3$ since there is a path of length 3 between a and e.

But also $(a,e)\in R^6$ since there is also another path of length 6 between a and e:
a,b,d,a,c,d,e

Theorem:
Let R be a relation on A, then there is a path of length n from a to b iff $(a, b) \in R^n$.

Proof: Use induction.

Basis step:
By definition there is a path of length 1 from a to b iff $(a, b) \in R$. Hence true for $n = 1$.

Theorem:

Let R be a relation on A, then there is a path of length n from a to b iff $(a, b) \in R^n$.

Proof: Use induction.

Basis step:

By definition there is a path of length 1 from a to b iff $(a, b) \in R$. Hence true for $n = 1$.

Inductive step: Assume it is true for some arbitrary fixed n. Show for $n+1$.

There is a path of length $n+1$ from a to b **iff**

$\exists c \in A$ s. t. there is a path of length 1 from a to c and a path of length n from c to b

Theorem:
Let R be a relation on A, then there is a path of length n from a to b iff $(a, b) \in R^n$.

Proof: Use induction.

Basis step:
By definition there is a path of length 1 from a to b iff $(a, b) \in R$. Hence true for $n = 1$.

Inductive step: Assume it is true for some arbitrary fixed n. Show for $n+1$.

There is a path of length $n+1$ from a to b **iff**

$\exists c \in A$ s. t. there is a path of length 1 from a to c and a path of length n from c to b
that is, $\exists c \in A$ such that $(a,c) \in R$ and $(c,b) \in R^n$ (by inductive hypothesis)

Theorem:

Let R be a relation on A, then there is a path of length n from a to b iff $(a, b) \in R^n$.

Proof: Use induction.

Basis step:

By definition there is a path of length 1 from a to b iff $(a, b) \in R$. Hence true for $n = 1$.

Inductive step: Assume it is true for some arbitrary fixed n. Show for $n+1$.

There is a path of length $n+1$ from a to b **iff**

$\exists c \in A$ s. t. there is a path of length 1 from a to c and a path of length n from c to b
that is, $\exists c \in A$ such that $(a,c) \in R$ and $(c,b) \in R^n$ (by inductive hypothesis)
which implies $(a, b) \in R^{n+1}$ (by definition of composite relation).

$\therefore$ There is a path of length $n+1$ from a to b iff $(a, b) \in R^{n+1}$

Transitive Closure:

Finding transitive closure is equivalent to determining vertices that are **connected** through a path.

Definition:
Let R be a relation on A.
Connectivity relation R^* consists of all pairs (a, b) s.t. there's a path between a and b in R.

Since R^n includes all the paths of length n by the previous theorem,

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

Transitive Closure:

Finding transitive closure is equivalent to determining vertices that are **connected** through a path.

Definition:
Let R be a relation on A.
Connectivity relation R^* consists of all pairs (a, b) s.t. there's a path between a and b in R.

Since R^n includes all the paths of length n by the previous theorem,

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

e.g.
Let R be a relation on the set of people in the world that contains (a,b) if a has met b.

R^2: ? if $(a, b) \in R^2$ then $\exists c$ s.t. $(a, c) \in R$ and $(c, b) \in R$

R^*: ? $(a, b) \in R^*$ if there is a sequence of people, starting with a and ending with b.

Theorem:

The transitive closure of a relation R equals to the connectivity relation R^*.

Theorem:

The transitive closure of a relation R equals to the connectivity relation R^*.

Proof:

We must show that, (i) R^* is transitive and (ii) any transitive relation that contains R contains also R^*.

Theorem:

The transitive closure of a relation R equals to the connectivity relation R^*.

Proof:

We must show that, (i) R^* is transitive and (ii) any transitive relation that contains R contains also R^*.

i. R^* is transitive?

If $(a, b) \in R^*$, there is a path from a to b.

If $(b, c) \in R^*$, there is a path from b to c.

$\therefore$ There is a path from from a to c, which means $(a, c) \in R^*$.

ii.

Theorem:

The transitive closure of a relation R equals to the connectivity relation R^*.

Proof:

We must show that, (i) R^* is transitive and (ii) any transitive relation that contains R contains also R^*.

i. R^* is transitive?

If $(a, b) \in R^*$, there is a path from a to b.

If $(b, c) \in R^*$, there is a path from b to c.

$\therefore$ There is a path from from a to c, which means $(a, c) \in R^*$.

ii. Let S be any transitive relation that contains R, i.e. $R \subseteq S$. Show $R^* \subseteq S$.

Theorem:

The transitive closure of a relation R equals to the connectivity relation R^*.

Proof:

We must show that, (i) R^* is transitive and (ii) any transitive relation that contains R contains also R^*.

i. R^* is transitive?

If $(a, b) \in R^*$, there is a path from a to b.

If $(b, c) \in R^*$, there is a path from b to c.

$\therefore$ There is a path from from a to c, which means $(a, c) \in R^*$.

ii. Let S be any transitive relation that contains R, i.e. $R \subseteq S$. Show $R^* \subseteq S$.

Since S is transitive, $S^n \subseteq S$ (by the theorem in Sec. 9.1)

Theorem:
The transitive closure of a relation R equals to the connectivity relation R^*.
Proof:
We must show that, (i) R^* is transitive and (ii) any transitive relation that contains R contains also R^*.

i. R^* is transitive?
If $(a, b) \in R^*$, there is a path from a to b.
If $(b, c) \in R^*$, there is a path from b to c.
$\therefore$ There is a path from from a to c, which means $(a, c) \in R^*$.

ii. Let S be any transitive relation that contains R, i.e. $R \subseteq S$. Show $R^* \subseteq S$.
Since S is transitive, $S^n \subseteq S$ (by the theorem in Sec. 9.1)

$$S^n \subseteq S \text{ and } S^* = \bigcup_{n=1}^{\infty} S^n \quad \Rightarrow \quad S^* \subseteq S$$

Theorem:

The transitive closure of a relation R equals to the connectivity relation R^*.

Proof:

We must show that, (i) R^* is transitive and (ii) any transitive relation that contains R contains also R^*.

i. R^* is transitive?

If $(a, b) \in R^*$, there is a path from a to b.

If $(b, c) \in R^*$, there is a path from b to c.

$\therefore$ There is a path from from a to c, which means $(a, c) \in R^*$.

ii. Let S be any transitive relation that contains R, i.e. $R \subseteq S$. Show $R^* \subseteq S$.

Since S is transitive, $S^n \subseteq S$ (by the theorem in Sec. 9.1)

$S^n \subseteq S$ and $S^* = \bigcup_{n=1}^{\infty} S^n \quad \Rightarrow \quad S^* \subseteq S$

Since $R \subseteq S$ (given), $R^* \subseteq S^*$

$\therefore$ $R^* \subseteq S$.

Thus any transitive relation S that contains R contains also R^*. ✵ Given R, how can we compute the connectivity relation R^*?

$$R^* = \bigcup_{n=1}^{\infty} R^n \text{ ?}$$

✵ Given R, how can we compute the connectivity relation R^*?

Lemma:

Let R be a relation in A and $|A| = n$. If there is a path from a to b in R, then one can always find a path from a to b with length not exceeding n.

✵ Given R, how can we compute the connectivity relation R^*?

Lemma:

Let R be a relation in A and $|A| = n$. If there is a path from a to b in R, then one can always find a path from a to b with length not exceeding n.

Proof:

Suppose there is a path $x_0, x_1, \ldots, x_m$ from $x_0 = a$ to $x_m = b$ with length m.

If $m > n$, then there are at least two vertices on this path, equal to each other $x_i = x_j$ such that $0 \leq i < j \leq m - 1$. (by the pigeonhole principle)

We can cut this circuit and form a new path

$$x_0, x_1, \ldots, x_i, x_{j+1}, \ldots, x_m$$

If we do the same for all such two vertices, we get a path of length $\leq n$.

✵ Given R, how can we compute the connectivity relation R^*?

<u>Lemma:</u>

Let R be a relation in A and $|A| = n$. If there is a path from a to b in R, then one can always find a path from a to b with length not exceeding n.

Hence by the Lemma,

$$R^* = \bigcup_{k=1}^{\infty} R^k = \bigcup_{k=1}^{n} R^k$$

Theorem:

Let $\mathbf{M}_R$ be zero-one matrix of R on a set A with n elements. Then the zero-one matrix representation of R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}^2_R \vee \mathbf{M}^3_R \vee \ldots \vee \mathbf{M}^n_R$$

e.g.

Let $R = \{(a, a), (a, c), (b, a), (c, a), (c, c)\}$. Find R^*.

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \mathbf{M}^2_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \mathbf{M}^3_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}^2_R \vee \mathbf{M}^3_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Algorithm for computation of connectivity relation:

Transitive closure ($\mathbf{M}_R$: zero-one matrix representation of R)

$\mathbf{A} = \mathbf{M}_R$
$\mathbf{B} = \mathbf{A}$
for (i=2; $i \le n$; i++) {
  $\mathbf{A} = \mathbf{A} \odot \mathbf{M}_R$
  $\mathbf{B} = \mathbf{B} \vee \mathbf{A}$
}
return $\mathbf{B}$

Note that transitive closure is identical to connectivity relation.

Algorithm for computation of connectivity relation:

Transitive closure ($\mathbf{M}_R$: zero-one matrix representation of R)

$\mathbf{A} = \mathbf{M}_R$
$\mathbf{B} = \mathbf{A}$
for (i=2; $i \leq n$; i++) {
    $\mathbf{A} = \mathbf{A} \odot \mathbf{M}_R$
    $\mathbf{B} = \mathbf{B} \vee \mathbf{A}$
}
return $\mathbf{B}$

Note: Although in general $S \circ R \neq R \circ S$, while computing powers of a relation, the order of compositions does not matter, hence

$$R^{n+1} = R^n \circ R = R \circ R^n \Rightarrow \mathbf{M}^{n+1}_R = \mathbf{M}^n_R \odot \mathbf{M}_R = \mathbf{M}_R \odot \mathbf{M}^n_R$$

Algorithm for computation of connectivity relation:

Transitive closure ($\mathbf{M}_R$: zero-one matrix representation of R)

$\mathbf{A} = \mathbf{M}_R$
$\mathbf{B} = \mathbf{A}$
for (i=2; $i \leq n$; i++) {
$\mathbf{A} = \mathbf{A} \odot \mathbf{M}_R$
$\mathbf{B} = \mathbf{B} \vee \mathbf{A}$
}
return $\mathbf{B}$

Complexity:

$\mathbf{A} \odot \mathbf{M}_R$: $(n + (n - 1))n^2$ operations

$\mathbf{B} \vee \mathbf{A}$: n^2 operations

$T(n) = (n - 1)\ (n^2(2n - 1) + n^2) = (n - 1)(2n^3)$

$\therefore$ $T(n)$ is $O(n^4)$. (Polynomial complexity) *e.g.* Let $(a, b) \in R$ if there is a non-stop flight from city a to b.

When is (a, b) in

R^2? If $\exists c$ s.t. $(a, c) \in R$, $(c, b) \in R$.

R^3? If $\exists c,d$ s.t. $(a, c) \in R$, $(c, d) \in R$, $(d, b) \in R$.

R^*? If it is possible to fly from a to b.

R^* can be computed using the algorithm of the previous slide.

9.5 Equivalence Relations

e.g. Consider the relation $R = \{(a, b) \mid a \equiv b \pmod 4\}$
R is **symmetric**, **transitive** and **reflexive**.

Hence we say, R is an **equivalence** relation.

9.5 Equivalence Relations

e.g. Consider the relation $R = \{(a, b) \mid a \equiv b \pmod 4\}$
R is **symmetric**, **transitive** and **reflexive**.

Hence we say, R is an **equivalence** relation.

What matters?

R divides (or partitions) the set of integers into four disjoint subsets:

$\{\ldots,-8,-4,0,4,8,\ldots\}$, $\{\ldots,-7,-3,1,5,9,\ldots\}$, $\{\ldots,-6,-2,2,6,10,\ldots\}$, $\{\ldots,-5,-1,3,7,11,\ldots\}$

where any two integers in a given subset is related with R, hence said to be "equivalent" to each other.

$(4,8) \in R$ hence 4 is equivalent to 8, and so is (1,5).

Definition: **Equivalence Relation**

If a relation is reflexive, symmetric and transitive then it is called an equivalence relation.

Equivalent elements: Two elements that are related by an equivalence relation.

- R (defined previously) is an equivalence relation, more specifically a "modular" equivalence relation.
- $(4,8) \in R$ hence 4 is equivalent to 8, and so is (1,5).

Another relation example that defines an equivalence between strings:

Let R be a relation on the set of strings :

$$R = \{(a, b) \mid L(a) = L(b)\},\ \text{where } L(x) \text{ is the length of string } x.$$

Another relation example that defines an equivalence between strings:

Let R be a relation on the set of strings :

$$R = \{(a, b) \mid L(a) = L(b)\}, \text{ where } L(x) \text{ is the length of string } x.$$

- R is reflexive since $\forall a \; L(a) = L(a)$ which implies that $(a, a) \in R$.

Another relation example that defines an equivalence between strings:

Let R be a relation on the set of strings :

$$R = \{(a, b) \mid L(a) = L(b)\},\ \text{where } L(x) \text{ is the length of string } x.$$

- R is reflexive since $\forall a\ \ L(a) = L(a)$ which implies that $(a, a) \in R$.
- R is symmetric since $\forall a,b\ \ (a, b) \in R \rightarrow L(a) = L(b) \rightarrow L(b) = L(a) \rightarrow (b, a) \in R$

Another relation example that defines an equivalence between strings:

Let R be a relation on the set of strings :

$$R = \{(a, b) \mid L(a) = L(b)\},$$ where $L(x)$ is the length of string x.

- R is reflexive since $\forall a \;\; L(a) = L(a)$ which implies that $(a, a) \in R$.
- R is symmetric since $\forall a,b \;\; (a, b) \in R \rightarrow L(a) = L(b) \rightarrow L(b) = L(a) \rightarrow (b, a) \in R$
- R is transitive since $\forall a,b,c \;\; (a, b) \in R \wedge (b, c) \in R \rightarrow L(a) = L(b) \wedge L(b) = L(c)$
  $\rightarrow L(a) = L(c) \rightarrow (a, c) \in R$

$\therefore$ R is an equivalence relation.

Another relation example that defines an equivalence between strings:

Let R be a relation on the set of strings :

$$R = \{(a, b) \mid L(a) = L(b)\}, \text{ where } L(x) \text{ is the length of string } x.$$

- R is reflexive since $\forall a \;\; L(a) = L(a)$ which implies that $(a, a) \in R$.
- R is symmetric since $\forall a,b \;\; (a, b) \in R \rightarrow L(a) = L(b) \rightarrow L(b) = L(a) \;\rightarrow (b, a) \in R$
- R is transitive since $\forall a,b,c \;\; (a, b) \in R \wedge (b, c) \in R \;\rightarrow\; L(a) = L(b) \;\wedge\; L(b) = L(c)$
  $\rightarrow\; L(a) = L(c) \;\rightarrow (a, c) \in R$

$\therefore$ R is an equivalence relation.

“discrete” is equivalent to “computer” with respect to R.

Another relation example that defines an equivalence between strings:

Let R be a relation on the set of strings :

$$R = \{(a, b) \mid L(a) = L(b)\}, \text{ where } L(x) \text{ is the length of string } x.$$

- R is reflexive
- R is symmetric
- R is transitive

$\therefore$ R is an equivalence relation.

"discrete" is equivalent to "computer" with respect to R.

- R divides (or partitions) the set of strings into disjoint subsets, where each subset contains all strings of the same length.

- Any two strings in a given subset are equivalent to each other (with respect to the relation).

e.g.
Relations on a set of people:

a) $\{(a, b) \mid a$ and b are at the same age$\}$

b) $\{(a, b) \mid a$ and b speak a common language$\}$

Are they equivalence relations?

e.g.
Relations on a set of people:

a) $\{(a, b) \mid a$ and b are at the same age$\}$ Yes

b) $\{(a, b) \mid a$ and b speak a common language$\}$ No

Are they equivalence relations?

Equivalence Classes:

e.g.

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

What is the equivalence class of 1 with respect to R?

1 is equivalent to $1-m$, 1, $1+m$, and so on.

Hence, the equivalence class of 1:

$$[1]_R = \{\ldots., 1-m, 1, 1+m, 1+2m, \ldots.\}$$

Definition: **Equivalence class**

Let R be an equivalence relation on A.

The set of all elements that are related to an element a of A is called the equivalence class of a:

$$[a]_R = \{ s \mid (a, s) \in R \}$$

e.g.

$$[1]_R = \{\ldots., 1-m, 1, 1+m, 1+2m, \ldots.\}$$

e.g. Consider the equivalence relation on the set of strings.
$R = \{(a, b) \mid L(a) = L(b)\}$, where $L(x)$ is the length of string x.

The equivalence class of the string "discrete" is the set of all strings with 8 characters.

Let S_n denote the set of all strings with n characters. Then the above equivalence relation partitions the set of all strings S into infinitely many disjoint and nonempty subsets, S_1, S_2, S_3, …

Equivalence Classes and Partitions:

Equivalence classes partition (or divide) a set into disjoint, nonempty subsets.

Proof: See Chapter 9.5 of your textbook, page 591 (7th edition).

Equivalence Classes and Partitions:

Equivalence classes partition (or divide) a set into disjoint, nonempty subsets.

e.g.

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

m equivalence classes:

$[0]_R$, $[1]_R$, ….., $[m-1]_R$

All are disjoint and form a partition.

Equivalence Classes and Partitions:

Equivalence classes partition (or divide) a set into disjoint, nonempty subsets.

e.g.

Relation R on a set of people: $\{(a, b) \mid a$ and b are at the same age$\}$

R partitions the set of people into equivalence classes (hence into nonempty disjoint subsets).

Each equivalence class is the set of people who are at the same age, for example $[a]_R$ is the set of people who are 18 years old (if a is 18 years old).

e.g.
Let R be a relation on the set of positive real-number pairs s.t.

$$((a, b), (c, d)) \in R \leftrightarrow ad = bc$$

Show that R is an equivalence relation.

e.g.

Let R be a relation on the set of positive real-number pairs s.t.

$$((a, b), (c, d)) \in R \leftrightarrow ad = bc$$

Show that R is an equivalence relation.

R is reflexive since $\forall a,b \quad ab = ab$, which implies that $((a, b), (a, b)) \in R$

e.g.
Let R be a relation on the set of positive real-number pairs s.t.

$$((a, b), (c, d)) \in R \leftrightarrow ad = bc$$

Show that R is an equivalence relation.

R is reflexive since $\forall a,b \quad ab = ab$, which implies that $((a, b), (a, b)) \in R$
R is symmetric since $\forall a,b,c,d \quad ((a, b), (c, d)) \in R \rightarrow ad = bc \rightarrow da = cb \rightarrow ((c, d), (a, b)) \in R$.

e.g.

Let R be a relation on the set of positive real-number pairs s.t.

$$((a, b), (c, d)) \in R \leftrightarrow ad = bc$$

Show that R is an equivalence relation.

R is reflexive since $\forall a,b \;\; ab = ab$, which implies that $((a, b), (a, b)) \in R$

R is symmetric since $\forall a,b,c,d \;\; ((a, b), (c, d)) \in R \to ad = bc \to da = cb$
$\to ((c, d), (a, b)) \in R$.

R is transitive since $\forall a,b,c,d,e,f \;\; ((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R \to (ad = bc$ and $cf = de) \to af = be \to ((a, b), (e, f)) \in R$

e.g.

Let R be a relation on the set of positive real-number pairs s.t.

$$((a, b), (c, d)) \in R \leftrightarrow ad = bc$$

Show that R is an equivalence relation.

R is reflexive since $\forall a,b \quad ab = ab$, which implies that $((a, b), (a, b)) \in R$

R is symmetric since $\forall a,b,c,d \quad ((a, b), (c, d)) \in R \rightarrow ad = bc \rightarrow da = cb$
$\rightarrow ((c, d), (a, b)) \in R$.

R is transitive since $\forall a,b,c,d,e,f \quad ((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R \rightarrow (ad = bc$ and $cf = de) \rightarrow af = be \rightarrow ((a, b), (e, f)) \in R$

Thus, equivalence classes for this relation partition the set of positive real number pairs into disjoint nonempty subsets such that

$[(a,b)]_R = \{(x,y) \mid x/y = a/b = c$, c is a positive real number$\}$

9.6 Partial Order Relations

We can use *relations* to order/sort elements of a set.

e.g. $S = \{1, 3, 4, 2, 5\}$

- $R = \{(a, b) \mid a \leq b\}$ is reflexive, anti-symmetric and transitive, thus it is a "partial order relation", and we can use it as a criterion to order elements of the set S:

  1, 2, 3, 4, 5 (ascending order)

9.6 Partial Order Relations

We can use *relations* to order/sort elements of a set.

e.g. $S = \{1, 3, 4, 2, 5\}$

- $R = \{(a, b) \mid a \leq b\}$ is reflexive, anti-symmetric and transitive, thus it is a "partial order relation", and we can use it as a criterion to order elements of the set S:

  1, 2, 3, 4, 5 (ascending order)

- $R = \{(a, b) \mid a \geq b\}$ is also reflexive, anti-symmetric and transitive, thus also a partial order relation, that defines a different criterion:

  5, 4, 3, 2, 1 (descending order)

9.6 Partial Order Relations

We can use *relations* to order/sort elements of a set.

e.g. $S = \{1, 3, 4, 2, 5\}$

- $R = \{(a, b) \mid a \leq b\}$ is reflexive, anti-symmetric and transitive, thus it is a "partial order relation", and we can use it as a criterion to order elements of the set S:

  1, 2, 3, 4, 5 (ascending order)

- $R = \{(a, b) \mid a \geq b\}$ is also reflexive, anti-symmetric and transitive, thus also a partial order relation, that defines a different criterion:

  5, 4, 3, 2, 1 (descending order)

- $R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (2,1), (2,3), (3,1), (1,4), (2,4), (3,4), (5,1), (5,4)\}$ is reflexive, anti-symmetric and transitive, thus a partial order relation, that yet defines another criterion:

  5, 2, 3, 1, 4 (some weird order)

Definition: **Partial Order Relation:**

A relation R on a set S is called partial order relation (or a partial ordering) iff it is reflexive, anti-symmetric and transitive.

A set S together with a partial ordering R is called a partially ordered set, or **poset**, and is denoted by (S, R).

e.g.

$R = \{(a, b) \mid a \leq b\}$ is a partial order on Z.

Hence (Z, R) is a poset.

e.g.

$R = \{(a, b) \mid a \mid b\}$ is a partial order on Z.

Definition: **Partial Order Relation:**
A relation R on a set S is called partial order relation (or a partial ordering) iff it is reflexive, anti-symmetric and transitive.

A set S together with a partial ordering R is called a partially ordered set, or **poset**, and is denoted by (S, R).

e.g.

$R = \{(a, b) \mid a \leq b\}$ is a partial order on Z.
Hence (Z, R) is a poset.

e.g.

$R = \{(a, b) \mid a \mid b\}$ is a partial order on Z.

A partial order relation defines what means being 'less than or equal to' *e.g.* $S = \{1, 3, 4, 2, 5\}$

- $R = \{(a, b) \mid a \leq b\}$ is a partial ordering:

  $1 \preccurlyeq 2 \preccurlyeq 3 \preccurlyeq 4 \preccurlyeq 5$ (ascending order)

- $R = \{(a, b) \mid a \geq b\}$ is a partial ordering:

  $5 \preccurlyeq 4 \preccurlyeq 3 \preccurlyeq 2 \preccurlyeq 1$ (descending order)

- $R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (2,1), (2,3), (3,1), (1,4), (2,4), (3,4), (5,1), (5,4)\}$ is a partial ordering:

  $5 \preccurlyeq 2 \preccurlyeq 3 \preccurlyeq 1 \preccurlyeq 4$

$a \preccurlyeq b$ means $(a, b) \in R$. (where R is a partial order) *e.g.* $S = \{1, 3, 4, 2, 5\}$

- $R = \{(a, b) \mid a \leq b\}$ is a partial ordering:
  $1 \prec 2 \prec 3 \prec 4 \prec 5$ (ascending order)

- $R = \{(a, b) \mid a \geq b\}$ is a partial ordering:
  $5 \prec 4 \prec 3 \prec 2 \prec 1$ (descending order)

- $R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (2,1), (2,3), (3,1), (1,4), (2,4), (3,4), (5,1), (5,4)\}$ is a partial ordering:
  $5 \prec 2 \prec 3 \prec 1 \prec 4$

$a \prec b$ means $(a, b) \in R$, but $a \neq b$.

We can write $1 \prec 2$ or $1 \preccurlyeq 2$ or $1 \preccurlyeq 1$ but not $1 \prec 1$

Comparable vs Incomparable

e.g.
Poset (Z^+, R) with $R = \{(a, b) \mid a \mid b\}$

$(3,6) \in R$, thus $3 \preccurlyeq 6$, hence 3 and 6 are **comparable**.

$(3,10) \notin R$, thus we cannot write $3 \preccurlyeq 10$ or $10 \preccurlyeq 3$, hence 3 and 10 are **incomparable**.

Definition:

The elements a, b of a poset $(S, \preccurlyeq)$ are called **comparable** if either $a \preccurlyeq b$ or $b \preccurlyeq a$. Otherwise they are called **incomparable**.

e.g.

Let $R = \{(a,a), (b,b), (c,c), (b,a), (b,c), (c,a)\}$ on $S = \{a, b, c\}$.

R is reflexive, anti-symmetric and transitive $\therefore$ R is a partial order $\therefore$ (S, R) is a poset.

All elements are comparable: $a \preccurlyeq a$ $b \preccurlyeq c$ $b \preccurlyeq a$

Definition:

The elements a, b of a poset $(S, \preccurlyeq)$ are called **comparable** if either $a \preccurlyeq b$ or $b \preccurlyeq a$. Otherwise they are called **incomparable**.

e.g.
Let $R = \{(a,a), (b,b), (c,c), (b,a), (b,c), (c,a)\}$ on $S = \{a, b, c\}$.
R is reflexive, anti-symmetric and transitive $\therefore$ R is a partial order $\therefore$ (S, R) is a poset.
All elements are comparable: $a \preccurlyeq a \quad b \preccurlyeq c \quad b \preccurlyeq a$

e.g.
Poset (Z^+, R) with $R = \{(a, b) \mid b \mid a\}$.
$1 \preccurlyeq 1 \quad 4 \preccurlyeq 2 \quad 10 \preccurlyeq 5$

We cannot write $2 \preccurlyeq 3$ or $3 \preccurlyeq 2$, hence 2 and 3 are incomparable.

Definition:

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a totally ordered set and $\preccurlyeq$ is called a **total order(ing)**.

e.g. Poset (Z^+, R) with $R = \{(a, b) \mid a \mid b\}$
R is **not** a total ordering since there are elements incomparable such as 5 and 6.

e.g. Poset (S, R) with $R = \{(a,a), (b,b), (c,c), (b,a), (b,c), (c,a)\}$ on $S = \{a, b, c\}$
R is a total ordering; every two elements are comparable.

Definition:

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a totally ordered set and $\preccurlyeq$ is called a **total order(ing)**.

e.g. Poset (Z^+, R) with $R = \{(a, b) \mid a \mid b\}$
R is **not** a total ordering since there are elements incomparable such as 5 and 6.

e.g. Poset (S, R) with $R = \{(a,a), (b,b), (c,c), (b,a), (b,c), (c,a)\}$ on $S = \{a, b, c\}$
R is a total ordering; every two elements are comparable.

e.g. $R = \{(a, b) \mid b \leq a\}$ on the set of integers.

R is a total ordering; every two elements are comparable, for instance $10 \preccurlyeq 5$.

e.g. $R_t = \{(a,b) \mid a$ is taller than b or $a = b\}$ on the set of all people.
?

e.g. $R_t = \{(a,b) \mid a$ is taller than or of the same height with $b\}$ on the set of all people.
?

Definition:

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a totally ordered set and $\preccurlyeq$ is called a **total order(ing)**.

e.g. Poset (Z^+, R) with $R = \{(a, b) \mid a \mid b\}$
R is **not** a total ordering since there are elements incomparable such as 5 and 6.

e.g. Poset (S, R) with $R = \{(a,a), (b,b), (c,c), (b,a), (b,c), (c,a)\}$ on $S = \{a, b, c\}$
R is a total ordering; every two elements are comparable.

e.g. $R = \{(a, b) \mid b \le a\}$ on the set of integers.

R is a total ordering; every two elements are comparable, for instance $10 \preccurlyeq 5$.

e.g. $R_t = \{(a,b) \mid a$ is taller than b or $a = b\}$ on the set of all people.
R_t is **not** a total ordering; two different people with the same height are not comparable.
e.g. $R_t = \{(a,b) \mid a$ is taller than or of the same height with $b\}$ on the set of all people.
?

Definition:

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a totally ordered set and $\preccurlyeq$ is called a **total order(ing)**.

e.g. Poset (Z^+, R) with $R = \{(a, b) \mid a \mid b\}$
R is **not** a total ordering since there are elements incomparable such as 5 and 6.

e.g. Poset (S, R) with $R = \{(a,a), (b,b), (c,c), (b,a), (b,c), (c,a)\}$ on $S = \{a, b, c\}$
R is a total ordering; every two elements are comparable.

e.g. $R = \{(a, b) \mid b \leq a\}$ on the set of integers.

R is a total ordering; every two elements are comparable, for instance $10 \preccurlyeq 5$.

e.g. $R_t = \{(a,b) \mid a$ is taller than b or $a = b\}$ on the set of all people.
R_t is **not** a total ordering; two different people with the same height are not comparable.
e.g. $R_t = \{(a,b) \mid a$ is taller than or of the same height with $b\}$ on the set of all people.
R_t is not even a partial ordering since it is not anti-symmetric.

Lexicographic (Alphabetic) Order

e.g. Comparing strings by $\preccurlyeq$ = {(a, b) | letter a appears before letter b in the alphabet or a = b}.

"that" $\prec$ "this"

why?

Lexicographic (Alphabetic) Order

e.g. Comparing strings by $\preccurlyeq$ = {(a, b) | letter a appears before letter b in the alphabet or a = b}.

“that” $\prec$ “this”

since

t = t, h = h and a $\prec$ i.

Lexicographic (Alphabetic) Order

To be able to compare *n*-tuples

$(a_1, a_2, \ldots, a_n)$?≺ $(b_1, b_2, \ldots, b_n)$,

we need *n* posets (hence *n* partial order relations):

$(A_1, \preccurlyeq_1), (A_2, \preccurlyeq_2), \ldots, (A_n, \preccurlyeq_n)$, where $a_1, b_1 \in A_1$; $a_2, b_2 \in A_2$ and so on.

Lexicographic (Alphabetic) Order

To be able to compare n-tuples

$$(a_1, a_2, \ldots, a_n) \ ?\prec \ (b_1, b_2, \ldots., b_n),$$

we need n posets (hence n partial order relations):

$(A_1, \preccurlyeq_1), (A_2, \preccurlyeq_2), \ldots., (A_n, \preccurlyeq_n)$, where $a_1, b_1 \in A_1$; $a_2, b_2 \in A_2$ and so on.

Then we can write

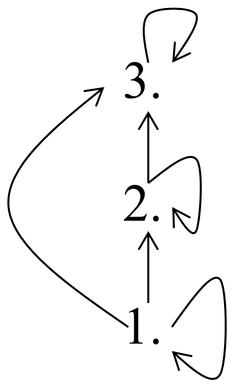
$$(a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots., b_n)$$

whenever

$a_1 \prec_1 b_1$ or $\exists i \ \ 0 < i < n$ such that $a_1 = b_1, \ldots., a_i = b_i$, and $a_{i+1} \prec_{i+1} b_{i+1}$

Hasse Diagrams:

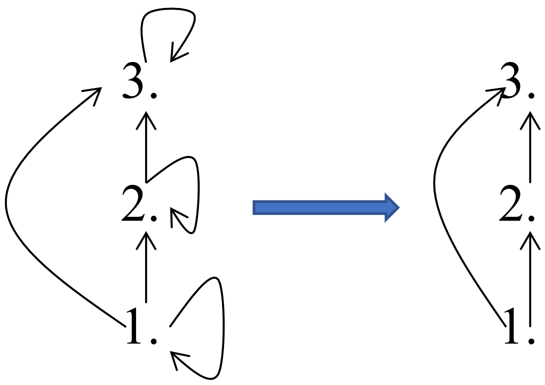
Let's consider the graph representation of $(\{1, 2, 3\}, \leq)$



Partial ordering

Hasse Diagrams:

Let's consider the graph representation of $(\{1, 2, 3\}, \leq)$

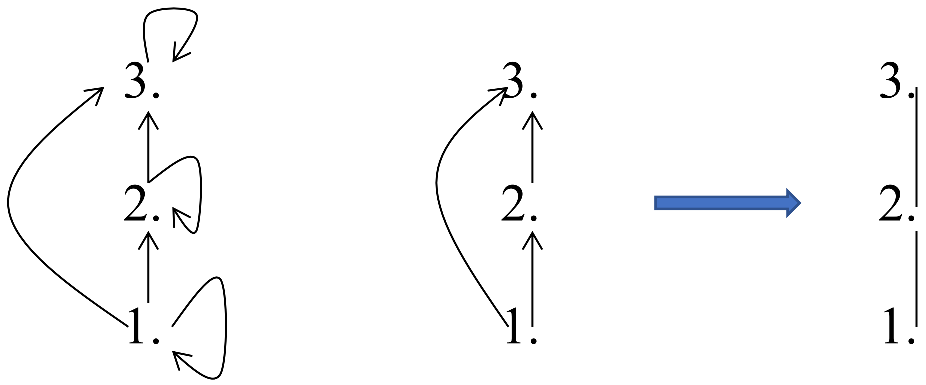


Partial ordering:

- Reflexive: Since each node has loop, no need to draw loops explicitly.

Hasse Diagrams:

Let’s consider the graph representation of $(\{1, 2, 3\}, \leq)$



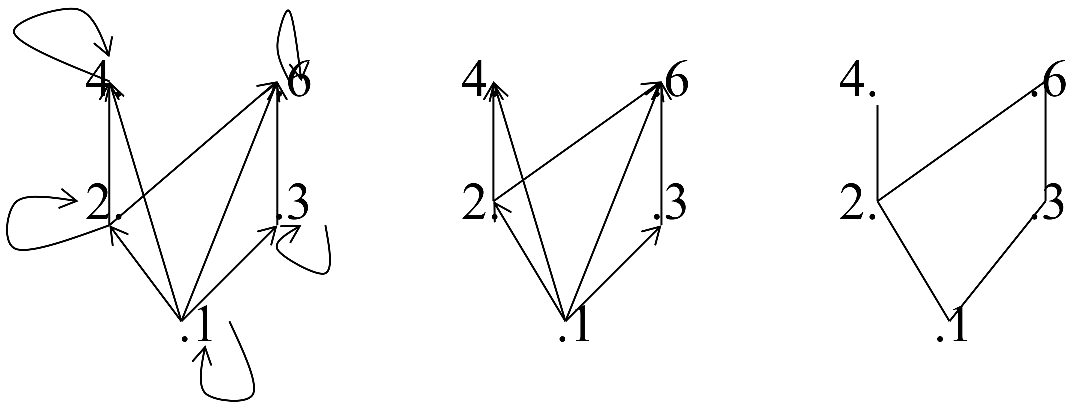
Hasse diagram

Partial ordering:

- Reflexive: Since each node has loop, no need to draw loops explicitly.
- Transitive: Since having (1, 2), (2, 3) means we have (1, 3), no need to draw the edge (1, 3) explicitly; same for other transitions.
- Anti-symmetric: No need to show directions, since we can assume all edges pointed upwards by convention.

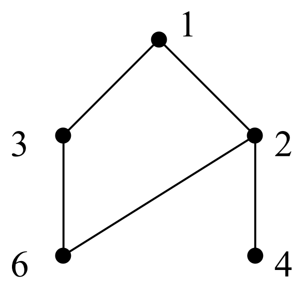
e.g.

Consider the partial ordering $\{(a, b) \mid a \mid b\}$ on $\{1, 2, 3, 4, 6\}$.



Hasse diagram

e.g.
Consider the partial ordering $\{(a, b) \mid b \mid a\}$ on $\{1, 2, 3, 4, 6\}$.



3 and 6 are comparable: $6 \preccurlyeq 3$

2 and 3 are not comparable.

e.g.

Consider the partial ordering

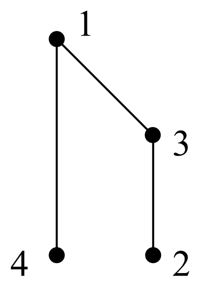
$R = \{(1, 1), (2,2), (3, 3), (4,4), (2, 1), (2, 3), (3, 1), (4,1)\}$ on $S = \{1, 2, 3, 4\}$.

Draw the Hasse Diagram.

e.g.
Consider the partial ordering

$R = \{(1, 1), (2,2), (3, 3), (4,4), (2, 1), (2, 3), (3, 1), (4,1)\}$ on $S = \{1, 2, 3, 4\}$.

Draw the Hasse Diagram.



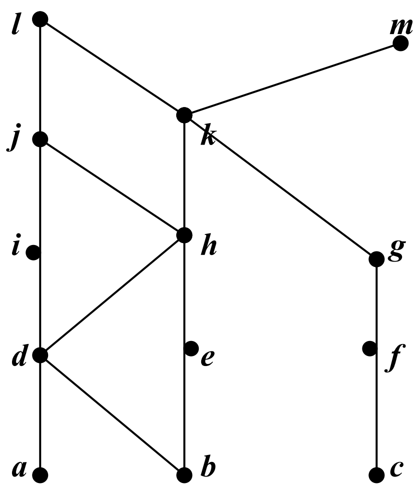
4,2 are minimal elements
1 is maximal element

Maximal and Minimal Elements:

An element a is a maximal in poset $(S, \preccurlyeq)$ if there is no $b \in S$ s.t. $a \prec b$.
(top elements in Hasse diagram)

An element a is a minimal in poset $(S, \preccurlyeq)$ if there is no $b \in S$ s.t. $b \prec a$.
(bottom elements in Hasse diagram)

e.g. (from textbook)



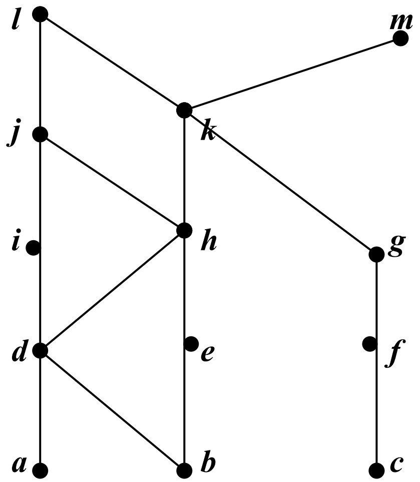
a) Maximal elements: ?
b) Minimal elements: ?

Maximal and Minimal Elements:

An element a is a maximal in poset $(S, \preccurlyeq)$ if there is no $b \in S$ s.t. $a \prec b$.
(top elements in Hasse diagram)

An element a is a minimal in poset $(S, \preccurlyeq)$ if there is no $b \in S$ s.t. $b \prec a$.
(bottom elements in Hasse diagram)

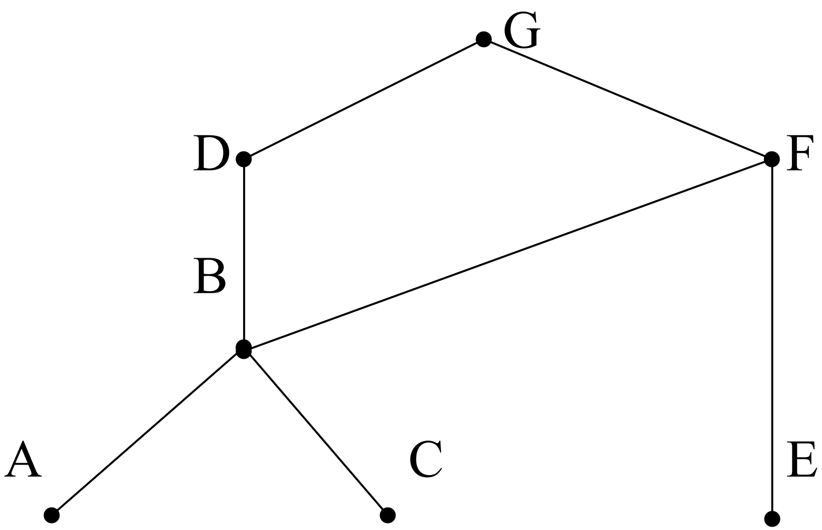
e.g. (from textbook)



a) Maximal elements: l, m
b) Minimal elements: a, b, c

Compatible Total Ordering

e.g. (from textbook)



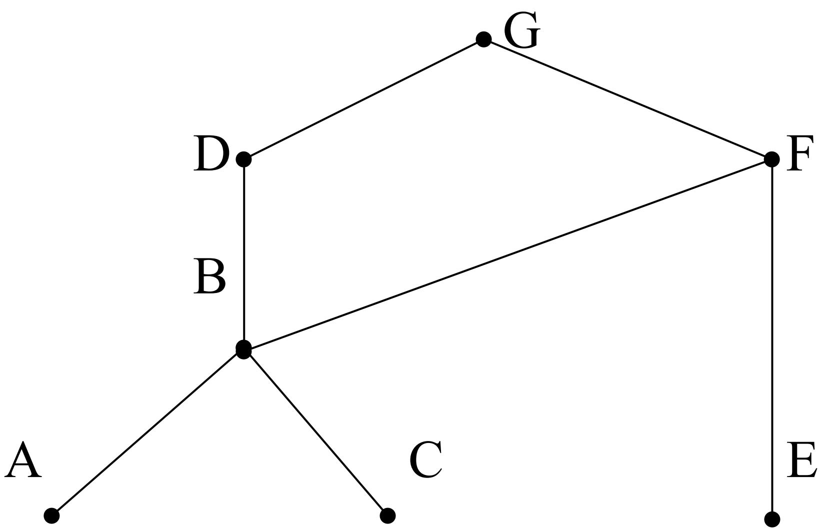
Hasse Diagram for scheduling seven tasks

Poset (S,R):

$S = \{A, B, C, D, E, F, G\}$
$R = \{(T_1, T_2) \mid \text{task } T_1 \text{ must precede } T_2 \text{ or } (T_1 = T_2), T_1,T_2 \in S\}$

Compatible Total Ordering

e.g.



Hasse Diagram for scheduling seven tasks

There are various compatible total orderings:

A, C, E, B, D, F, G

or

A , C , B, E, F, D, G

or

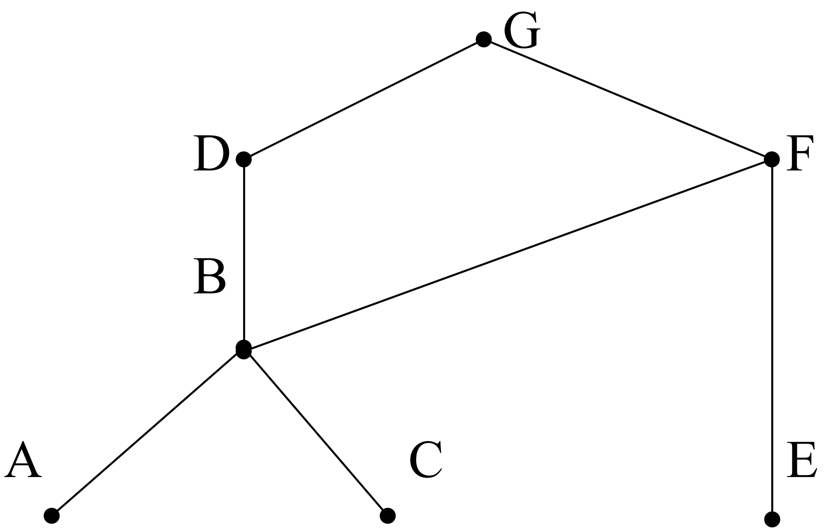
.

.

.

Compatible Total Ordering

e.g.



Hasse Diagram for scheduling seven tasks

There are various compatible total orderings:

A, C, E, B, D, F, G

or

A , C , B, E, F, D, G

or

.

.

.

- Total ordering because all the elements in the set are ordered.
- Each of these orderings is compatible with the partial order relation.
- We can find these compatible total orders in general by applying the topological sorting algorithm.
- See the next slide.

Topological Sorting Algorithm

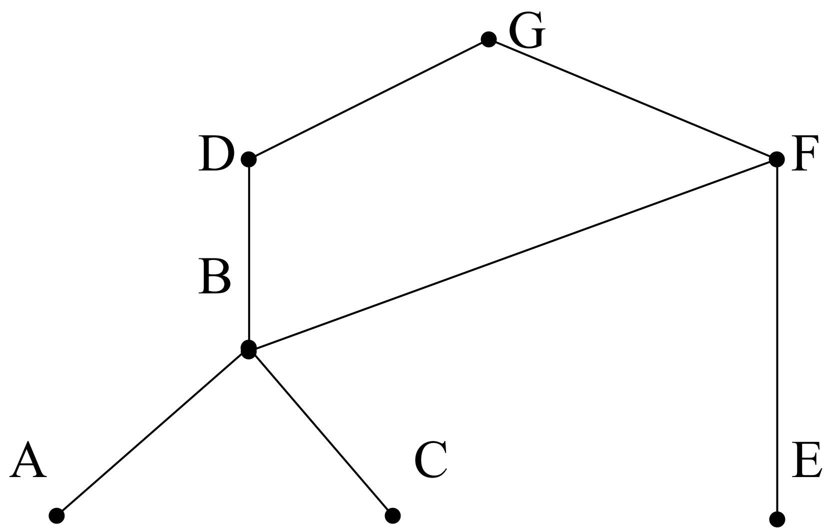
A **compatible total ordering** can be constructed with a partial ordering R:

Topo-sort (S,R: finite poset)
$k = 1$
while $S \neq \varnothing$ {
a_k = minimal element of S
$S = S - \{a_k\}$
$k = k + 1$
}

$\{a_1, a_2, \ldots, a_n\}$ is a compatible total ordering of S (compatible with relation R).

How to apply toplogical sorting algorithm to the Hasse diagram below:

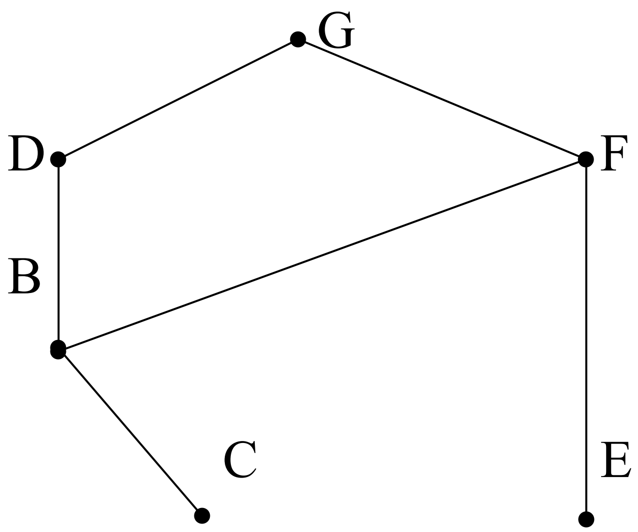
e.g.



Minimal elements are A, C and E; pick one, say A.

How to apply toplogical sorting algorithm to the Hasse diagram below:

e.g.

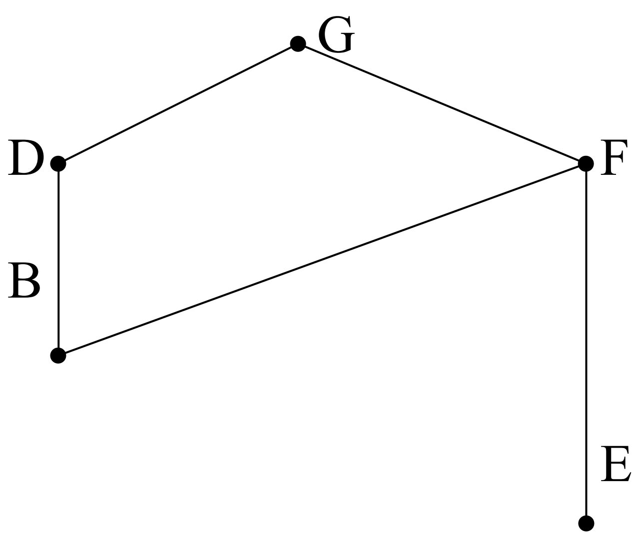


A,

Remove A and its connections.
Minimal elements are then C and E; pick one, say C.

How to apply toplogical sorting algorithm to the Hasse diagram below:

e.g.

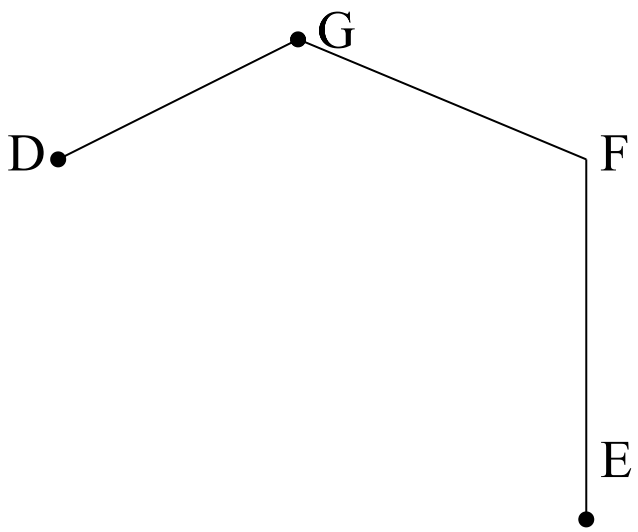


A, C,

Remove C and its connections.
Minimal elements are then B and E; pick one, say B.

How to apply toplogical sorting algorithm to the Hasse diagram below:

e.g.

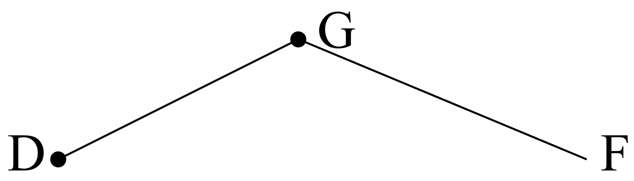


A, C, B,

Remove B and its connections.
Minimal elements are then D and E; pick one, say E.

How to apply toplogical sorting algorithm to the Hasse diagram below:

e.g.

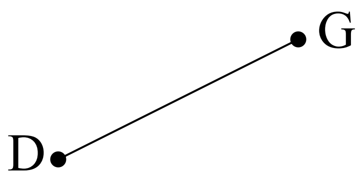


A, C, B, E,

Remove E and its connections.
Minimal elements are then D and F; pick one, say F.

How to apply toplogical sorting algorithm to the Hasse diagram below:

e.g.



A, C, B, E, F,

Remove F and its connections.
The only minimal element is then D; pick D.

How to apply toplogical sorting algorithm to the Hasse diagram below:

e.g.

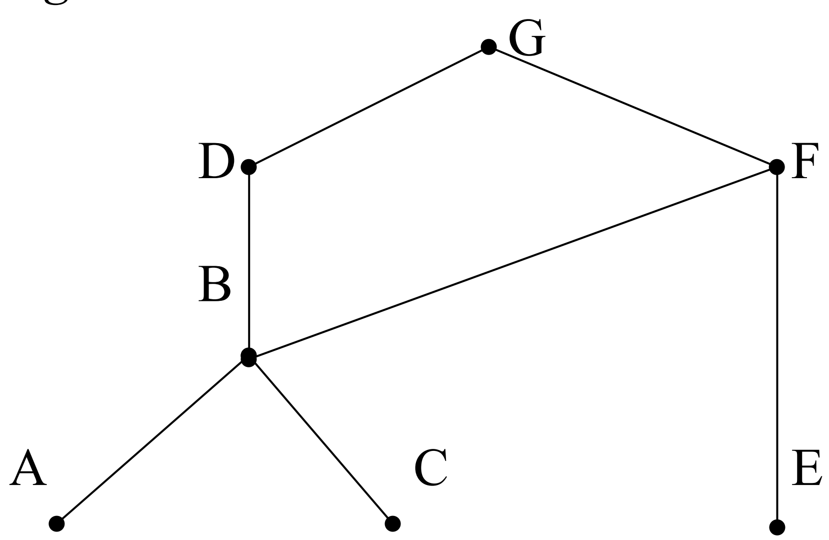
G

A, C, B, E, F, D,

Remove D and its connections.
The only element is then G; pick G.

How to apply toplogical sorting algorithm to the Hasse diagram below:

e.g.



We get A, C, E, B, D, F, G as a compatible total ordering.
Depending on the choices you make to pick a minimal element, you may end up with different compatible total ordering alternatives, all valid and respecting the partial ordering relation.

e.g. (from textbook) Draw the Hasse Diagram and find a compatible total ordering for the given poset.

$$R = \{(a, b) \mid a \mid b\}$$

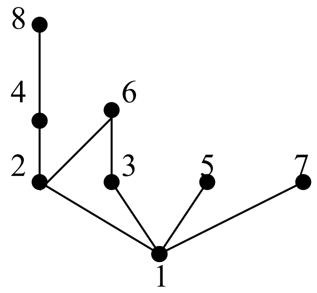
a) $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

b) $A = \{1, 2, 3, 6, 12, 24, 36, 48\}$

e.g. (from textbook) Draw the Hasse Diagram and find a compatible total ordering for the given poset.

$R = \{(a, b) \mid a \mid b\}$

a) $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$



One possible compatible total ordering:
1, 7, 3, 5, 2, 4, 8, 6

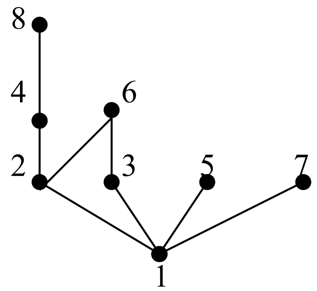
Another: 1, 2, 3, 5, 7, 4, 8, 6

b) $A = \{1, 2, 3, 6, 12, 24, 36, 48\}$

e.g. (from textbook) Draw the Hasse Diagram and find a compatible total ordering for the given poset.

$R = \{(a, b) \mid a \mid b\}$

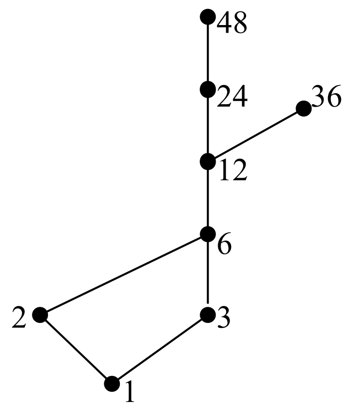
a) $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$



One possible compatible total ordering:
1, 7, 3, 5, 2, 4, 8, 6

Another: 1, 2, 3, 5, 7, 4, 8, 6

b) $A = \{1, 2, 3, 6, 12, 24, 36, 48\}$



One possible compatible total ordering:
1, 3, 2, 6, 12, 24, 36, 48

e.g. (from textbook) Draw Hasse Diagram for the inclusion relation R on the power set $P(S)$ where $S = \{a, b, c, d\}$.

$R = \{(S_1, S_2) \mid S_1 \subseteq S_2,\ S_1, S_2 \in P(S)\}$