Comp-106, Fall 2019, HW#4

(Most of the problems below are very similar to or the same as the questions from the relevant chapters of your textbook.).

Please provide formal justification for all your answers to the following questions in order to get full credit.

1. Use mathematical induction to prove the following Lemma (from your textbook and lecture slides):

If p is a prime and $p \mid a_1 a_2 \ldots a_n$, where a_i is an integer for $i = 1, 2, 3, \ldots, n$, then $p \mid a_i$ for some integer i.

2. Prove that $f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1}$ when n is a positive integer.

3. Show that $f_n < (5/3)^n$, where is f_n the nth Fibonacci number. (Hint: Use **strong induction**).

4. Device a recursive algorithm (i.e., write down a pseudocode for a recursive method/function) that computes the sum of the first n integers. Then prove **by induction** that your algorithm produces the correct output.

5. Device two algorithms (one **recursive**, the other **iterative**) to find the nth term of the sequence defined by $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ and $a_n = a_{n-1} + 2a_{n-2} + a_{n-3}$, for $n = 3,4,5,\ldots$
Write down the complexity functions of both algorithms. Is the recursive or the iterative algorithm for finding this sequence more efficient? Explain your answer by giving big-O, big-Θ or big-Ω estimates (whichever appropriate) for both algorithms.

6. Prove by using **strong** induction that the recursive algorithm that you found in Exercise 5 is correct.

7. Solve the recurrence relations below, together with the initial conditions given, using their characteristic equations:

       i) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$
       ii) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = -3$
       iii) $a_n = 4a_{n-1}$ for $n \geq 1$, $a_0 = 3$

8. Solve the recurrence relation $a_n = 4a_{n-1}$ for $n \geq 1$, $a_0 = 3$ by using **generating functions**. Note that this is the same recurrence as Question 7 part iii, but this time you're asked to solve it by using generating functions.