Analysis and Numerical Simulation of Hybrid Differential-Algebraic Equations

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Mathematics for key technologies

Outline

Analysis and numerical solution of hybrid systems described by differential-algebraic equations (DAEs)

Applications

- electronic circuits (different device models for different frequencies), cooperation with NEC
- mechanical systems (robot manipulators, automatic gear-boxes), cooperation with Daimler AG

- systems from biological or chemical engineering,
- traffic systems, which operate different depending on delays.

Definition

A hybrid DAE system is a set of nonlinear DAEs

$$F^l(t, x^l, \dot{x}^l) = 0, \quad F^l : D_l \times \mathbb{R}^{n_l} \times \mathbb{R}^{n_l} \to \mathbb{R}^{m_l}, \quad l \in \mathbb{M},$$

with sufficiently smooth functions F^l for each mode in $D_l = \bigcup_i [\alpha_i, \beta_i)$, where

- ▷ each mode l has a number of transitions $j \in J^l$, with switching functions

$$g_j^l(t, x^l, \dot{x}^l) \geq 0,$$

- ▷ the successor mode k is determined by the mode allocation function

$$S^l : J^l \to \mathbb{M}, \text{ such that } S^l(j) = k,$$

- ▷ and each transition j has a transition function

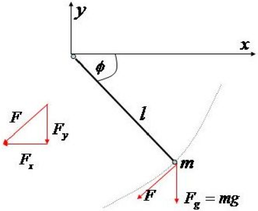
$$T_j^l(x^l(\beta_i)) = x^k(\alpha_{i+1}), \quad \beta_i = \alpha_{i+1}.$$

In general, also controls u^l, outputs y^l, parameters p, and uncertainties w in each mode. In this talk only analysis and numerics!

Tangentially accelerated pendulum

$$m\ddot{x} = -2x\lambda + F_x$$
$$m\ddot{y} = -2y\lambda - mg + F_y \quad \text{(mode 1)}$$
$$0 = x^2 + y^2 - l^2$$

$$m\ddot{x} = 0 \quad \text{(mode 2)}$$
$$m\ddot{y} = -mg$$



$J^1 = \{1\},\ S^1(1) = 2,\ g_1^1 = F_c - \dot{x}^2 - \dot{y}^2$
$T_1^1(x, y, \lambda) = [x, y]^T$ at $t = \beta_1 = \alpha_2$.

Why DAEs ?

DAEs form a common framework for analysis, simulation and control of coupled dynamical systems.

- ▷ Automatic modular modelling (Simulink/Modellica) leads to DAEs.
- ▷ Space discretized conservation laws lead to DAEs.
- ▷ Simulator coupling leads to discrete DAEs.
- ▷ Control problems are DAEs.
- ▷ Robust/optimal control leads to DAE boundary value problems.

Classical approach

Solve for algebraic equations (minimal coordinates).

Problems:

- Variables without physical meaning.
- Loss of modularity.
- Numerical solution drifts off constraints after a few time steps.

Black-box modelling with DAEs

Modelling becomes easy, all problems are pushed into the numerics. The numerical methods cannot handle this!
Problems:

- Numerical simulation does not allways work, stability and convergence problems (e.g. Simulink) !
- Consistent initialization difficult.
- The resulting nonlinear system may be unsolvable even if the DAE is solvable, (see later example).
- Numerical drift-off phenomenon due to unresolved hidden constraints.
- Model reduction difficult.
- Classical control approaches difficult (non-proper transfer functions).

Today several packages use computer algebra (Modellica, Dymola) to turn back to ODE.

What do we do?

- Component- and network-based remodelling.
- Strangeness-free (index 1 formulation) keeping all open and hidden constraints, interfaces, and all variables.
- Strangeness-free formulation of control problems, continuous, discrete and hybrid.
- Black-box-software `GELDA, GENDA` for small systems.
- Special software for automatic MBS `GEOMS`.
- Special software for circuits.
- H_∞-controller design, model reduction, optimal control for strangeness-free models.

$$E\dot{x} = Ax + f(t), \quad t \in \mathbb{I},$$

where $E, A \in \mathbb{C}^{l,n}$ and $f \in C(\mathbb{I}, \mathbb{C}^{l})$.
Consider scaling from the left and changes of basis with nonsingular matrices.

$$PEQ\dot{\tilde{x}} = PAQ\tilde{x} + Pf(t), \; \tilde{x}(t_0) = \tilde{x}_0.$$

Definition

Two pairs of matrices $(E_i, A_i), \; i = 1, 2$, are called *(strongly) equivalent* if there exist invertible matrices $P \in \mathbb{C}^{l,l}, Q \in \mathbb{C}^{n,n}$ with $E_2 = PE_1Q, \; A_2 = PA_1Q$.

Theorem

Weierstraß/Kronecker 1890-1896 *For every pair* $E, A \in \mathbb{C}^{l,n}$ *there exist nonsingular* $P \in \mathbb{C}^{l,l}, Q \in \mathbb{C}^{n,n}$ *such that*
$P(\lambda E - A)Q = \mathrm{Diag}\,(L_{\epsilon_1}, \ldots, L_{\epsilon_p}, M_{\eta_1}, \ldots, M_{\eta_q}, J_{\rho_1}, \ldots, J_{\rho_v}, N_{\sigma_1}, \ldots, N_{\sigma_w})$,

$$J_{\rho_j} = \lambda \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} - \begin{bmatrix} \lambda_j & 1 & & \\ & & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}, \quad M_{\eta_j} = \lambda \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} - \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix},$$

$$N_{\sigma_j} = \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} - \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}, \quad L_{\epsilon_j} = M_{\epsilon_j}^T$$

Definition

- ▷ A matrix pencil $\lambda E - A,\ E, A \in \mathbb{C}^{\ell,n}$, is called *regular* if $\ell = n$ and if

$$P(\lambda) = \det(\lambda E - A)$$

  does not vanish identically, otherwise *singular*.
- ▷ The size ν_d of the largest nilpotent (N)-blocks in the KCF is called the *differentiation-index* of $\lambda E - A$.

Theorem

Campbell 1982 Consider a linear constant coefficient system with regular $\lambda E - A$ and let $f \in C^{\nu_d}(\mathbb{I}, \mathbb{C}^n)$.
Then the system is solvable and every consistent initial condition fixes a unique solution.

$$E(t)\dot{x}(t) = A(t)x(t) + f(t),\ x(t_0) = x_0,\ t, t_0 \in \mathbb{I}.$$

Scaling from the left and changes of basis with nonsingular matrix functions.

$$P(t)E(t)Q(t)\dot{\tilde{x}} = (P(t)A(t)Q(t) - P(t)E(t)\dot{Q}(t))\tilde{x} + P(t)f(t),\ \tilde{x}(t_0) = \tilde{x}_0.$$

Definition

Two pairs of matrix functions $(E_i(t), A_i(t))$ in $\mathbb{C}^{l,n}$ are called *globally equivalent* if there exist $P \in C(\mathbb{I}, \mathbb{C}^{l,l})$ and $Q \in C^1(\mathbb{I}, \mathbb{C}^{n,n})$, $P(t), Q(t)$ nonsingular for all $t \in \mathbb{I}$ such that

$$[E_2(t), A_2(t)] = P(t)[E_1(t), A_1(t)] \begin{bmatrix} Q(t) & -\dot{Q}(t) \\ 0 & Q(t) \end{bmatrix}.$$

Regularity of the pencil at time t and the d-index at time t are not invariants under global equivalence.

A system that is uniformly regular but not uniquely solvable. The system

$$\begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x(t), \quad t \in \mathbb{R}$$

is uniformly regular and of uniform d-index $\nu_d = 2$ but

$$x(t) = c(t) \begin{bmatrix} t \\ 1 \end{bmatrix}$$

is a solution for all $c \in C^1(\mathbb{R}, \mathbb{C})$.

A system that is uniformly singular but uniquely solvable. The system

$$\begin{bmatrix} 0 & 0 \\ 1 & -t \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -1 & t \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix},$$

is uniformly singular, because the pencil is singular for all t.
But the system has the unique solution

$$\begin{bmatrix} f_1 + tf_2 - t\dot{f}_1 \\ f_2 - \dot{f}_1 \end{bmatrix}$$

independent of any initial condition.

Local version of global equivalence

Definition

Two pairs of matrices

$$(E_i, A_i),\ E_i, A_i \in \mathbb{R}^{l,n}, \quad i = 1, 2$$

are called *locally equivalent* if there exist matrices $P \in \mathbb{C}^{l,l}$, $Q, R \in \mathbb{C}^{n,n}$ with P, Q nonsingular such that

$$[E_2, A_2] = P[E_1, A_1] \begin{bmatrix} Q & -R \\ 0 & Q \end{bmatrix}.$$

By Hermite interpolation there always exists a function $Q(t)$ such that at any point $\hat{t}$ one has $Q(\hat{t}) = Q$ and $\dot{Q}(\hat{t}) = R$.

Theorem

Kunkel/M. 1994 *Let* $E, A \in \mathbb{C}^{l,n}$ *and*

$$\begin{array}{ll} (a) & T \text{ basis of kernel } E \\ (b) & Z \text{ basis of Co-range } E = \text{kernel } \; E^* \\ (c) & T' \text{ basis of Co-kernel } E = \text{kernel } \; E^* \\ (d) & V \text{ basis of Co-range } (Z^* AT). \end{array}$$

Then, the quantities (convention rank $\emptyset = 0$*)*

$$\begin{array}{lll} (a) & r = \text{rank } \; E & \text{(rank)} \\ (b) & a = \text{rank } (Z^* AT) & \text{(algebraic part)} \\ (c) & s = \text{rank } (V^* Z^* AT') & \text{(strangeness)} \\ (d) & d = r - s & \text{(differential part)} \\ (e) & v = l - r - a - s & \text{(redundant part)} \end{array}$$

are invariant under the local equivalence transformation.

Furthermore, (E, A) is locally equivalent to the canonical form:

$$\begin{matrix} s \\ d \\ a \\ s \\ v \end{matrix} \left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right).$$

Note: Eigenvalues are not invariants of this normal form.

Applying the local canonical form for all t we get functions

$$r, a, s : \mathbb{I} \to \{0, \ldots, l\}.$$

Theorem

Kunkel/M. 1994 Let E, A be sufficiently smooth and let r, a, s be constant in $\mathbb{I}$. Then $(E(t), A(t))$ is globally equivalent to a pair of matrix functions of the form

$$\begin{matrix} s \\ d \\ a \\ s \\ v \end{matrix} \left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t) \\ 0 & 0 & 0 & A_{24}(t) \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right).$$

$$\begin{array}{llll} (a) & \dot{x}_1 & = A_{12}(t)x_2 + A_{14}(t)x_4 + g_1(t) \\ (b) & \dot{x}_2 & = A_{24}(t)x_4 + g_2(t) \\ (c) & 0 & = x_3 + g_3(t) \\ (d) & 0 & = x_1 + g_4(t) \\ (e) & 0 & = g_5(t). \end{array}$$

Insert the derivative of (d) in (a), which becomes an algebraic equation. This gives

$$\begin{array}{c} s \\ d \\ a \\ s \\ v \end{array} \left(\begin{bmatrix} \mathbf{0} & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t) \\ 0 & 0 & 0 & A_{24}(t) \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right),$$

for which we can again compute characteristic values r, a, s, d, v.

Proceeding inductively we get a sequence of pairs of matrix functions $(E_i(t), A_i(t))$ and integers $r_i, a_i, s_i, d_i, v_i,\ i \in \mathbb{N}_0$, which we assume to be constant in $\mathbb{I}$.

We start with $(E_0(t), A_0(t)) = (E(t), A(t))$, and then $(E_{i+1}(t), A_{i+1}(t))$ is derived from $(E_i(t), A_i(t))$ by bringing it into canonical form and inserting the derivative of (d) into (a). The procedure stops after finitely many steps.

Definition

The number μ of steps is called the *strangeness-index or s-index* μ. If $\mu = 0$, then the system is called *strangeness-free*.

Theorem

Kunkel/M. 1994 Let the s-index μ be well–defined for $(E(t), A(t))$ and let $f \in C^{\mu}(\mathbb{I}, \mathbb{C}^{l})$. Then the system is equivalent to a DAE in normal form

$$\begin{array}{rcll} \dot{x}_1(t) & = & A_{13}(t)x_3(t) + f_1(t), & d_\mu \text{ equations,} \\ 0 & = & x_2(t) + f_2(t), & a_\mu \text{ equations,} \\ 0 & = & f_3(t), & v_\mu \text{ equations,} \end{array}$$

where the inhomogeneity is determined by $f^{(0)}, \ldots, f^{(\mu)}$.

- *The problem is solvable if and only if $f_3(t) \equiv 0$.*
- *An initial condition is consistent if and only if in addition $x_2(t_0) = -f_2(t_0)$ holds.*
- *The problem is uniquely solvable if again in addition we have $u_\mu = n - d_\mu - a_\mu = 0$.*
- *Otherwise, we can choose $x_3 \in C(\mathbb{I}, \mathbb{C}^{u_\mu})$ arbitrarily (control).*

What do we learn from the canonical form

- The algebraic approach is essential for the theoretical understanding of DAEs, in particular in the hybrid setting.
- The approach allows to do bifurcation analysis.
- The points where ranks change are a superset of the set of *critical or switching points*.
- But, it cannot be used for numerical methods or the design of controllers, since one would need derivatives of computed transformation matrices.
- Numerical computation even in the strangeness-free case is very expensive.

Replacing $\dot{x}$ in $\mathcal{F}(t, x, \dot{x}) = 0$ by finite difference operators like `implicit Euler` or `BDF` in general does not work!

- The resulting system of nonlinear equations may not be solvable, even if the system has a unique solution. (Example 2).
- The convergence order of the finite difference method may be reduced by up to μ orders.
- The finite difference method may diverge.
- Even if all goes well, the numerical solution drifts off from the hidden constraints, i.e., one gets physically meaningless results.
- The approach cannot be applied for control problems.

For numerical methods and for the design of controllers, we use derivative arrays (Campbell 1989).
We assume that derivatives of original functions are available or can be obtained via computer algebra or automatic differentiation.

Linear case: We put $E(t)\dot{x} = A(t)x + f(t)$ and its derivatives up to order μ into a large DAE

$$M_k(t)\dot{z}_k = N_k(t)z_k + g_k(t), \quad k \in \mathbb{N}_0$$

for $z_k = (x, \dot{x}, \ldots, x^{(k)})$.

$$M_2 = \begin{bmatrix} E & 0 & 0 \\ A - \dot{E} & E & 0 \\ \dot{A} - 2\ddot{E} & A - \dot{E} & E \end{bmatrix}, \; N_2 = \begin{bmatrix} A & 0 & 0 \\ \dot{A} & 0 & 0 \\ \ddot{A} & 0 & 0 \end{bmatrix}, \; z_2 = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix}.$$

Numerically construction of canonical form

Theorem

Kunkel/M. 1996 Under some constant rank assumptions, for every linear DAE there exist integers μ, a, d and v such that from the derivative array of level μ we obtain (via orthogonal projection) a numerically computable
strangeness-free form

$$\begin{array}{rcll} \hat{E}_1(t)\dot{x} & = & \hat{A}_1(t)x + \hat{f}_1(t), & d \text{ equations} \\ 0 & = & \hat{A}_2(t)x + \hat{f}_2(t), & a \text{ equations} \\ 0 & = & \hat{f}_3(t), & v \text{ equations} \end{array}$$

where $\hat{A}_1 = Z_1^T A$, $\hat{f}_1 = Z_1^T f$, and $\hat{f}_2 = Z_2^T g_\mu$, $\hat{f}_3 = Z_3^T g_\mu$.
The partitioning is the same as in the canonical form

$$\begin{array}{rcll} \dot{x}_1(t) & = & A_{13}(t)x_3(t) + f_1(t), & d \text{ equations} \\ 0 & = & x_2(t) + f_2(t), & a \text{ equations} \\ 0 & = & f_3(t), & v \text{ equations.} \end{array}$$

Derivative arrays, nonlinear problems

Analogous approach for $F(t,x,\dot{x})=0$ yields derivative array:

$$0 = F_k(t,x,\dot{x},\ldots,x^{(k+1)}) = \begin{bmatrix} F(t,x,\dot{x}) \\ \frac{d}{dt}F(t,x,\dot{x}) \\ \cdots \\ \frac{d^k}{dt^k}F(t,x,\dot{x}) \end{bmatrix}.$$

We set

$$\begin{aligned} M_k(t,x,\dot{x},\ldots,x^{(k+1)}) &= F_{k;\dot{x},\ldots,x^{(k+1)}}(t,x,\dot{x},\ldots,x^{(k+1)}), \\ N_k(t,x,\dot{x},\ldots,x^{(k+1)}) &= -(F_{k;x}(t,x,\dot{x},\ldots,x^{(k+1)}),0,\ldots,0), \\ z_k &= (t,x,\dot{x},\ldots,x^{(k+1)}). \end{aligned}$$

Theorem

Kunkel/M. 2002 Under some constant rank assumptions and if $\mathbf{L} = F_\mu^{-1}(\{0\}) \neq \emptyset$*, then there exist locally integers* μ*, a, d and v such that for the derivative array of level* μ *we have that the solution set* $\mathbf{L}$ *forms a (smooth) manifold of dimension* $(\mu+2)n+1-r$*.*
The DAE can locally be transformed (by application of the implicit function theorem) to a reduced DAE of the form

$$\begin{aligned} \dot{x}_1 &= G_1(t, x_1, x_3), && \text{(d differential equations)},\\ x_2 &= G_2(t, x_1, x_3), && \text{(a algebraic equations)},\\ 0 &= 0 && \text{(v redundant equations)}. \end{aligned}$$

The variables x_3 *represent undetermined components (controls).*

- Consistent initial values are obtained by solving $F_\mu(t_0, x, \dot{x}, \dots, x^{(\mu+1)}) = 0$ at t_0 for the algebraic variable $(x, \dot{x}, \dots, x^{(\mu+1)})$.
- For the integration of the DAE, e.g. with BDF methods, the system

$$\begin{aligned} F_\mu(t_0 + h, x, \dot{x}, \dots, x^{(\mu+1)}) &= 0, \\ \tilde{Z}_1^T F(t_0 + h, x, D_h x) &= 0 \end{aligned}$$

  is solved for $(x, \dot{x}, \dots, x^{(\mu+1)})$.
- Here, $\tilde{Z}_1$ denotes a suitable approximation of Z_1 which projects onto the d differential equations at the desired solution, and

$$D_h x_i = \frac{1}{h} \sum_{l=0}^{k} \alpha_l x_{i-l},$$

  is the discretization by BDF.

Theorem

Kunkel/M. 2002 Under the assumtions of the local existence theorem, the occurring Jacobians of the system have full row rank at the solution provided the step-size h is sufficiently small and the approximation $\tilde{Z}_1$ is sufficiently good.

- Simplified Gauss-Newton method can be used to solve the nonlinear systems at every integration step.
- The order and convergence properties are the same as for ODEs.
- The method can be implemented by using local orthogonal projections.
- However, the projections may be expensive.

Several productions codes are available.

- Production code GELDA Kunkel/M./Rath/Weickert 1998 (linear variable coefficients), uses BDF and Runge-Kutta discretization.
- Production code GENDA (nonlinear regular), Kunkel/M./Seufer 2002 based on BDF.
- Matlab code SOLVEDAE (nonlinear), Kunkel/Mehrmann/Seidel 2005.
- Special multi-body code GEOMS Steinbrecher 2006.
- Circuit codes, joint with NEC, Bächle, Ebert, 2006.

Recall hybrid DAE systems

$$F^l(t, x^l, \dot{x}^l) = 0, \quad F^l : D_l \times \mathbb{R}^{n_l} \times \mathbb{R}^{n_l} \to \mathbb{R}^{m_l}, \quad l \in \mathbb{M},$$

with sufficiently smooth functions F^l for each mode in $D_l = \bigcup_i [\alpha_i, \beta_i)$, where

- each mode l has a number of transitions $j \in J^l$, with switching functions

$$g_j^l(t, x^l, \dot{x}^l) \geq 0,$$

- the successor mode k is determined by the mode allocation function

$$S^l : J^l \to \mathbb{M}, \text{ such that } S^l(j) = k,$$

- and each transition j has a transition function

$$T_j^l(x^l(\beta_i)) = x^k(\alpha_{i+1}), \quad \beta_i = \alpha_{i+1}.$$

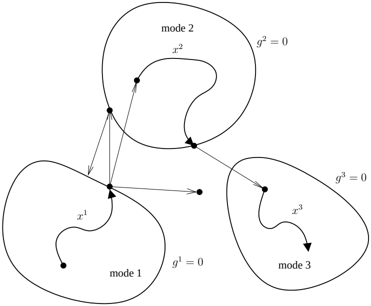
Transition Behavior at Switch Points

The switching functions define transition boundaries $\Gamma_j^l = \{g_j^l(t, x^l, \dot{x}^l) = 0\}$.

Behavior at transition boundary:

- ▷ regular switching,
- ▷ non-regular switching,
- ▷ chattering

  $\Longrightarrow$ sliding modes.

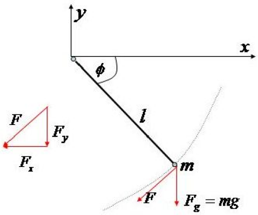


After a switch

- ▷ there could be changes in the dimension, structure, index or characteristic values,
- ▷ the state has to be transferred to the new mode in a consistent way (consistent reinitialization).

Tangentially accelerated pendulum:

$$m\ddot{x} = -2x\lambda + F_x$$
$$m\ddot{y} = -2y\lambda - mg + F_y \quad \text{(mode 1)}$$
$$0 = x^2 + y^2 - l^2$$



$$m\ddot{x} = 0 \quad \text{(mode 2)}$$
$$m\ddot{y} = -mg$$

$J^1 = \{1\}$, $S^1(1) = 2$, $g_1^1 = F_c - (\dot{x}^2 + \dot{y}^2)$ $T_1^1(x, y, \lambda) = [x, y]^T$ at $\beta_1 = \alpha_2$.

In Mode 1 we have $\mu = 2$, $d = 2$, $a = 1$, in Mode 2 we have $\mu = 0$, $d = 3$, $a = 0$.

For the index reduction we use the nonlinear derivative arrays in each mode:

- The derivative array $\mathcal{F}_k^l$ of level k in mode $l \in \mathbb{M}$ is given by

$$0 = \mathcal{F}_k^l(t, x^l, \dot{x}^l, \dots, x^{l(k+1)}) = \begin{bmatrix} F^l(t, x^l, \dot{x}^l) \\ \frac{d}{dt} F^l(t, x^l, \dot{x}^l) \\ \vdots \\ \frac{d^k}{dt^k} F^l(t, x^l, \dot{x}^l) \end{bmatrix}.$$

- We set

$$\mathcal{M}_k(t, x^l, \dot{x}^l, \dots, x^{l(k+1)}) = \mathcal{F}^l_{k;\dot{x}^l,\dots,x^{l(k+1)}}(t, x^l, \dot{x}^l, \dots, x^{l(k+1)}),$$
$$\mathcal{N}_k(t, x^l, \dot{x}^l, \dots, x^{l(k+1)}) = -(\mathcal{F}^l_{k;x^l}(t, x^l, \dot{x}^l, \dots, x^{l(k+1)}), 0, \dots, 0),$$
$$z_k^l = (t, x^l, \dot{x}^l, \dots, x^{l(k+1)}).$$

Under some constant rank assumptions, locally we get integers μ_l, r_l, a_l, d_l and v_l and we assume that the solution set

$$\mathbb{L}^l_{\mu_l} = \{(t, x^l, \ldots, x^{l(\mu_l+1)}) \in \mathbb{R}^{(\mu_l+2)n_l+1} \mid \mathcal{F}^l_{\mu_l}(t, x^l, \ldots, x^{l(\mu_l+1)}) = 0\}$$

is not empty .

Definition

For a hybrid DAE system the maximal strangeness index μ_{max} is defined as

$$\mu_{max} = \max_{l \in \mathbb{M}} \{\mu_l\}.$$

A hybrid DAE system is called strangeness-free if $\mu_{max} = 0$.

The extracted (strangeness-free) problem is given by

$$\hat{F}^l(t, x^l, \dot{x}^l) = \begin{bmatrix} \hat{F}^l_1(t, x^l, \dot{x}^l) \\ \hat{F}^l_2(t, x^l) \end{bmatrix} = \begin{bmatrix} (Z^l_1)^T F^l(t, x^l, \dot{x}^l) \\ \mathcal{F}^l_{\mu_l}(t, x^l, \ldots, x^{l(\mu_1+1)}) \end{bmatrix} = 0.$$

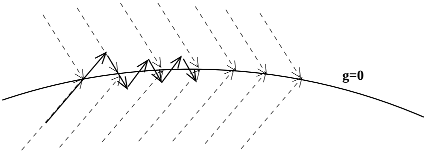
- We obtain information about the constraint manifold in each mode.
- Thus, consistent initial values can be obtained by solving the system

$$\mathcal{F}^{l}_{\mu_l}(\alpha_i, x^l, \dot{x}^l, \ldots, x^{l(\mu_l+1)}) = 0$$

  at the switch point α_i for $(x^l, \dot{x}^l, \ldots, x^{l(\mu_l+1)})$.
- We use the Gauß-Newton method started with a sufficiently good initial guess $(\tilde{x}^l, \ldots, \tilde{x}^{l(\mu_l+1)})$ to solve this system in a least square sense.
- As the Jacobians have full row rank, we have local quadratic convergence.
- We can fix arbitrary initial values for the differential variables, whereas initial values for the algebraic variables have to be computed consistently.

Sliding Mode Simulation

Chattering behavior can be approximated by sliding motion.

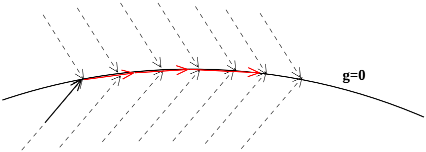


- We locally compute reduced systems

$$\dot{x}_1^l = \mathcal{L}^l(t, x_1^l), \qquad \dot{x}_1^k = \mathcal{L}^k(t, x_1^k),$$
$$x_2^l = \mathcal{R}^l(t, x_1^l), \qquad x_2^k = \mathcal{R}^k(t, x_1^k),$$

- the DAE in sliding motion is ($d_l = d_k$!)

$$\dot{x}_1 = \alpha\mathcal{L}^l(t, x_1) + (1-\alpha)\mathcal{L}^k(t, x_1),$$
$$x_2 = \tilde{\mathcal{R}}(t, x_1),$$
$$0 = g(t, x_1, x_2).$$



The system is augmented with appropriate algebraic constraints $x_2 = \tilde{\mathcal{R}}(t, x_1)$ to force the solution onto a specific constraint manifold.

The Sliding Condition

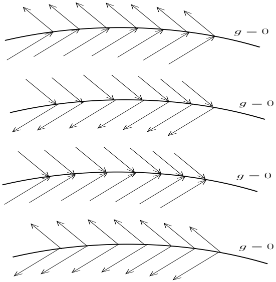
- The sliding condition is given by

$$\underbrace{\langle \frac{\partial}{\partial x_1^l} g_j^l(t, x_1^l, x_2^l), \mathcal{L}_\Gamma^l(t, x_1^l)\rangle}_{\mathcal{L}_N^l} < 0 \quad \text{and} \quad \underbrace{\langle \frac{\partial}{\partial x_1^k} g_j^k(t, x_1^k, x_2^k), \mathcal{L}_\Gamma^k(t, x_1^k)\rangle}_{\mathcal{L}_N^k} > 0.$$

- The normal projections onto the switching surface can be approximated by

$$\mathcal{L}_N^l \approx \frac{1}{\delta} g_j^l(t, x_1^l + \delta \mathcal{L}^l(t, x_1^l), x_2^l), \quad \mathcal{L}_N^k \approx \frac{1}{\delta} g_j^k(t, x_1^l + \delta \mathcal{L}^k(t, x_1^k), x_2^k).$$

1. If $\mathcal{L}_N^l > 0$ and $\mathcal{L}_N^k > 0$, the system switches from mode l to mode k.
2. If $\mathcal{L}_N^l < 0$ and $\mathcal{L}_N^k < 0$, the system switches from mode k to mode l.
3. If $\mathcal{L}_N^l < 0$ and $\mathcal{L}_N^k > 0$, the sliding condition is satisfied.
4. If $\mathcal{L}_N^l > 0$ and $\mathcal{L}_N^k < 0$, inconsistent switching.



- For the numerical integration of the DAE in mode $l \in \mathbb{M}$ from t_{i-1} to $t_i = t_{i-1} + h$ solve the nonlinear system

$$\mathcal{F}^l_{\mu_l}(t_i, x^l_i, \dot{x}^l_i, \ldots, x_i^{l(\mu_l+1)}) = 0,$$
$$\tilde{Z}_1^T F^l(t_i, x^l_i, D_h x^l_i) = 0$$

  for $(x^l_i, \dot{x}^l_i, \ldots, x_i^{l(\mu_l+1)})$, where $\tilde{Z}_1$ denotes an approximation of Z_1.
- The differential operator D_h denotes a BDF or Runge-Kutta method.
- The event time $t^\star$ is determined with a modified secant method as the root of the switching function, i.e.

$$g^l_j(t^\star, x^l(t^\star), \dot{x}^l(t^\star)) = 0, \quad \text{for some } j \in J^l.$$

- The solution of the system at a switch point $x^l(t^\star)$ as well as $\dot{x}^l(t^\star)$ are determined by interpolation (using the collocation polynomials):
  - for s-stage Runge-Kutta methods the interpolant has order s $(s-1)$,
  - for BDF methods of order k the interpolant has order k $(k-1)$.

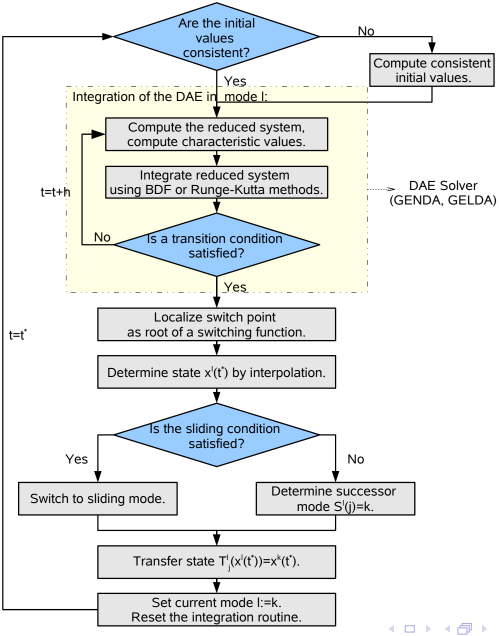
We have embedded several DAE solvers in a hybrid code.

- the code GELDA (Kunkel/Mehrmann/Rath/Weikert 1998) for over- and underdetermined linear variable coefficients DAEs (uses BDF and Runge-Kutta discretization),
- the code GENDA (Kunkel/Mehrmann/Seufert 2002) for general nonlinear DAEs (uses BDF discretization).

Additionally the following codes are currently incorporated.

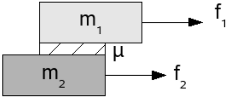
- the special multibody code GEOMS (Steinbrecher 2006) based on Runge-Kutta methods,
- electrical circuit codes,

Hybrid Mode Controller



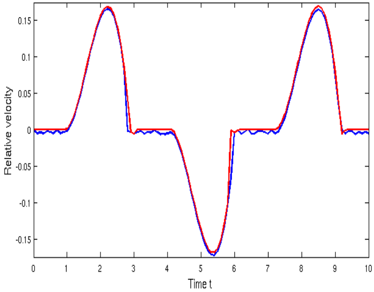
A simple numerical Example

$$\left.\begin{aligned} m_1\ddot{x}_1 &= f_1 - \mu|N|sgn(\dot{x}_1 - \dot{x}_2), \\ m_2\ddot{x}_2 &= f_2 + \mu|N|sgn(\dot{x}_1 - \dot{x}_2). \end{aligned}\right\} \text{ mode 1(2): } v_{rel} > (<) 0$$



$$\left.\begin{aligned} m_1\ddot{x}_1 &= f_1 + \lambda - \mu|N|, \\ m_2\ddot{x}_2 &= f_2 - \lambda + \mu|N|, \\ 0 &= \dot{x}_1 - \dot{x}_2. \end{aligned}\right\} \text{ (sliding mode, } v_{rel} \approx 0)$$

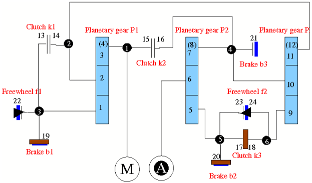
Solved with GELDA with $m_1 = m_2 = 1$, $f_1 = sin(t)$, $f_2 = 0$, $|N| = 1$, $\mu = 0.4$, $v_c = 0.007$, initial values $[x_{1,0}, x_{2,0}, \dot{x}_{1,0}, \dot{x}_{2,0}] = [1,\ 1,\ 0,\ 0]$ and $RTOL = ATOL = 10^{-4}$, $\mathbb{I} = [0, 10]$.



1. Solved without sliding.
2. Solved with sliding mode.

	No. steps	Switch points
1	4833	368
2	2709	89

New automatic gearbox Daimler AG

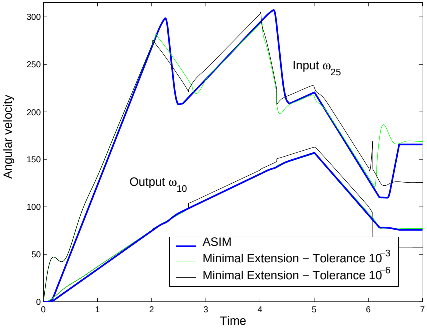


The model has in each mode the form of a mechanical multibody system

$$
\begin{aligned}
\dot{p} &= v \\
R\dot{v} &= f(p, v) - g_p^T(p)\lambda \\
0 &= g(p)
\end{aligned}
$$

and has between 70 and 100 variables in the different modes. Chattering occurs if the freewheels are included.

Comparison with Daimler in-house solver ASIM.



Summary and Current work

Done

- Modelling and analysis of hybrid DAE systems.
- Index reduction for hybrid systems.
- Consistent re-initialization after switching.
- Numerical treatment of chattering behavior. (Sliding modes).

To Do

- Use of specific structures to make the approach efficient (reduced derivative arrays, minimal extension).
- Incorporation of further DAE solvers (specially adapted for multibody systems, circuit equations, ...)
- Feedback control of hybrid systems
- Optimal and robust control.

Thank you for your attention!

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