KOÇ UNIVERSITY
Phys 301
Statistical Mechanics Instr: Alkan Kabakçıoğlu TA: Duygu Can

HW #2 SOLUTIONS

Q1. (Kittel&Kroemer 2.1) The question asks us to calculate the energy U and $\partial^2\sigma/\partial U^2$ for an ideal gas whose multiplicity function is given by $g(U) = CU^{3N/2}$.

(a) The (dimensionless) entropy σ is defined as the logarithm of the multiplicity and the inverse of temperature is the partial derivative of the entropy w.r.t. the energy. Then,

$$\begin{aligned} g(U) &= CU^{3N/2} \\ \Rightarrow \quad \sigma &= \ln g(U) = \ln C + \frac{3N}{2}\ln U \\ \Rightarrow \quad \frac{1}{\tau} &= \left.\frac{\partial\sigma}{\partial U}\right)_N = \frac{3N}{2}\cdot\frac{1}{U} \\ \Rightarrow \quad U &= \frac{3N}{2}\tau \end{aligned}$$

(b)

$$\left.\frac{\partial^2\sigma}{\partial U^2}\right)_N = -\frac{3N}{2}\cdot\frac{1}{U^2} < 0\ .$$

The negative sign of the derivative ensures that the energy of the system always increases with increasing temperature.

Q2. (Kittel&Kroemer 2.2) The energy of a state with spin excess 2s is given by $U = -2smB \Rightarrow s = -\frac{U}{2mb}$. Using Eq. (40) we find:

$$\begin{aligned} \sigma(s) &\simeq \ln g(N,0) - 2s^2/N \\ \Rightarrow \quad \sigma(U) &= \sigma_0 - U^2/2m^2B^2N \\ \Rightarrow \quad \frac{1}{\tau} &= \left.\frac{\partial\sigma}{\partial U}\right)_{N,B} = -\frac{U}{m^2B^2N} \\ \Rightarrow \quad U &= -m^2B^2N/\tau = -MB \\ \Rightarrow \quad \frac{M}{Nm} &= \frac{mB}{\tau} \end{aligned}$$

This result is known as the **Curie Law**: the magnetization of a paramagnet increases linearly with the applied uniform field and is inversely proportional to temperature.

Q3.(Kittel&Kroemer 2.3)

(a) The multiplicity function for N harmonic oscillators was found in Chapter 1. Using Eq. (1.55) we can find the entropy of the system as follows:

$$
\begin{aligned}
g(N,n) &= \frac{(N+n-1)!}{n!(N-1)!} \\
\Rightarrow \qquad \sigma &= \ln g(N,n) = \ln(N+n-1)! - \ln n! - \ln(N-1)! \\
&\simeq \ln(N+n)! - \ln n! - \ln N! \\
&= (N+n)\ln(N+n) - n\ln n - N\ln N \\
&= (N+n)\ln\left[N\left(1+\frac{n}{N}\right)\right] - n\ln\left(N\cdot\frac{n}{N}\right) - N\ln N \\
&= N\left[\ln\left(1+\frac{n}{N}\right)\left(1+\frac{n}{N}\right) - \frac{n}{N}\ln\frac{n}{N}\right]
\end{aligned}
$$

(b) Using $U = n\hbar\omega$ and $U_0 = N\hbar\omega$, we can replace n/N in the equation above with U/U_0 which leads to:

$$
\begin{aligned}
\sigma(U,N) &= N\left[\left(1+\frac{U}{U_0}\right)\ln\left(1+\frac{U}{U_0}\right) - \frac{U}{U_0}\ln\frac{U}{U_0}\right] \\
\Rightarrow \qquad \frac{1}{\tau} &= \left.\frac{\partial\sigma}{\partial U}\right)_N = \frac{N}{U_0}\left[\ln\left(1+\frac{U}{U_0}\right) - \ln\frac{U}{U_0}\right]
\end{aligned}
$$

Solving this equation for U gives:

$$
\begin{aligned}
e^{U_0/N\tau} &= \frac{U_0}{U} + 1 \\
\Rightarrow \qquad U &= \frac{U_0}{e^{U_0/N\tau} - 1} = \frac{N\hbar\omega}{e^{\hbar\omega/\tau} - 1}\ .
\end{aligned}
$$