1.1 Problem 4.1

Question

Show that the number of photons \( \sum < s_n > \) in equilibrium at temperature \( \tau \) in a cavity of volume \( V \) is

\[
N = 2.404 \pi^2 V (\tau/\hbar c)^3
\]  
(1.1)

From (23) the entropy is \( \sigma = (4 \pi^2 V/45)(\tau/\hbar c)^3 \), whence \( \sigma/N \simeq 3.602 \). It is believed that the total number of photons in the universe is \( 10^8 \) larger than the total number of nucleons(protons, neutrons). Because both entropies are of the order of the respective number of particles (see Eq. 3.76), the photons provide the dominant contribution to the universe, although the particles dominate the total energy. We believe that the entropy of the photons is essentially constant, so that the entropy of the universe is approximately constant with time.

Solution

\[
N = \sum_n < s_n > \]
(1.2)

\[
= \sum_n \frac{1}{e^{\hbar \omega_n/\tau} - 1}
\]  
(1.3)

where \( \omega_n = n \pi c/L \) and \( n = \sqrt{n_x^2 + n_y^2 + n_z^2} \). The sum over \( n_x, n_y, n_z \) is replaced by an integral over the volume element \( dn_x dn_y dn_z \) as follows,

\[
\sum (...) = 2 \times \frac{1}{8} \int_0^{\infty} 4 \pi n^2 dn(...)
\]  
(1.4)

Here the factor 2 arises since there are two independent polarizations of the electromagnetic field, and there is also a factor of \( \frac{1}{8} = \frac{1}{2^3} \) since only the positive octant of the space is involved. Then,

\[
N = \sum_n \frac{1}{e^{\hbar \omega_n/\tau} - 1}
\]
(1.5)

\[
\approx \pi \int_0^{\infty} dn \frac{n^2}{e^{\hbar \pi c/L \tau} - 1}
\]  
(1.6)

Setting \( u = \hbar n \pi c/L \tau \) we get,

\[
N = \pi^4 \left( \frac{L \tau}{\hbar \pi c} \right)^3 \int_0^{\infty} du \frac{u^2}{e^u - 1}
\]
(1.7)

\[
= \pi^{-2} V \left( \frac{\tau}{\hbar c} \right)^3 \Gamma(3) \zeta(3)
\]  
(1.8)

\[
\approx 2.404 \pi^{-2} V \left( \frac{\tau}{\hbar c} \right)^3
\]  
(1.9)
While taking the integral we have utilized,

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}dx}{e^x - 1}$$  \hspace{1cm} (1.10)

where $\Gamma$ is the Gamma function and $\zeta$ is the Riemann $\zeta$-function (http ://en.wikipedia.org/wiki/Riemann_zeta_function).

### 1.2 Problem 4.5

**Question**

Calculate the temperature of the surface of the Earth on the assumption that as a black body in thermal equilibrium it reradiates as much thermal radiation it receives from the Sun. Assume also that the surface of the Earth is at a constant temperature over the day-night cycle. Use $T_\odot = 5800\,\text{K}; \; R_\odot = 7 \times 10^{10}\,\text{cm}$; and the Earth-Sun distance of $D_{ES} = 1.5 \times 10^{13}\,\text{cm}$.

**Solution**

Total energy that sun produces,

$$U_\odot = J_\odot S_\odot = \sigma T_\odot^4 4\pi R_\odot^2$$  \hspace{1cm} (1.11)

$$= \sigma T_\odot^4 4\pi R_\odot^2$$  \hspace{1cm} (1.12)

Flux on Earth,

$$J_E = \frac{U}{4\pi D_{ES}^2} = \sigma T_\odot^4 \left(\frac{R_\odot}{D_{ES}}\right)^2$$  \hspace{1cm} (1.13)

$$= \sigma T_\odot^4 \left(\frac{R_\odot}{D_{ES}}\right)^2$$  \hspace{1cm} (1.14)

Energy received by Earth,

$$U_E = \pi R_E^2 J_E$$  \hspace{1cm} (1.15)

Energy re-radiated from Earth,

$$U_E = 4\pi R_E^2 \sigma T_E^4$$  \hspace{1cm} (1.16)

Equating these two energies and solving for $T_E$ we get,

$$T_E = \sqrt{\frac{R_\odot}{2D_{ES}}} T_\odot$$  \hspace{1cm} (1.17)

$$\approx 280\,\text{K}$$  \hspace{1cm} (1.18)
1.3 Problem 4.9

Question

Consider a transmission line of length $L$ on which electromagnetic waves satisfy the one-dimensional wave equation $v^2 \frac{\partial^2 E}{\partial x^2} = \frac{\partial^2 E}{\partial t^2}$, where $E$ is an electric field component. Find the heat capacity of the photons on the line, when in thermal equilibrium at temperature $\tau$. The enumeration of modes proceeds in the usual way for one dimension: take the solutions as standing waves with zero amplitude at each end of the line.

Solution

Wave equation in one dimension is,

$$v^2 \frac{\partial^2 E}{\partial x^2} = \frac{\partial^2 E}{\partial t^2} \quad (1.19)$$

Assume solution in the separable form, $E = X(x)T(t)$. Substitute this form of the solution into the wave equation.

$$v^2 \frac{\partial^2 X}{\partial x^2} T = X \frac{\partial^2 T}{\partial t^2} \quad (1.20)$$

Dividing both parts by $X(x)T(t)$ we get,

$$\frac{v^2}{X} \frac{\partial^2 X}{\partial x^2} = \frac{1}{T} \frac{\partial^2 T}{\partial t^2} \quad (1.21)$$

Notice that LHS of the equation above depends only on $x$ and the RHS of the equation depends only on $t$. Then they must both equal to a constant, say $-\omega^2$ (The negative sign is chosen to satisfy the BC’s on both ends). Then the solution of the wave equation in one dimension is,

$$E = E_0 \sin(\omega t) \sin(\omega x/v) \quad (1.22)$$

Since the wave is confined to $(0, L)$, $\omega_n = n\pi v/L$.

The total energy is,

$$U = \sum_n <s_n> \hbar \omega_n \quad (1.23)$$

$$= \sum_n \frac{\hbar \omega_n}{e^{\hbar \omega_n/\tau} - 1} \quad (1.24)$$

$$= \sum_n \frac{\hbar n\pi v/L}{e^{\hbar n\pi v/L\tau} - 1} \quad (1.25)$$

$$\approx \frac{L \tau^2}{\hbar \pi v} \int_0^\infty du \frac{uu}{e^u - 1} \quad (1.26)$$
\[
L = \frac{L\pi^2}{\hbar\pi v}\Gamma(2)\zeta(2) \\
= \frac{L\pi^2 \pi^2}{\hbar\pi v 6} \\
= \frac{L\pi\tau^2}{\hbar 6v}
\]

(1.27)

(1.28)

(1.29)

The heat capacity is,

\[
C_L = \frac{\partial U}{\partial T}\bigg|_L
\]

\[
= \frac{L\pi\tau}{3\hbar v}
\]

(1.30)

(1.31)