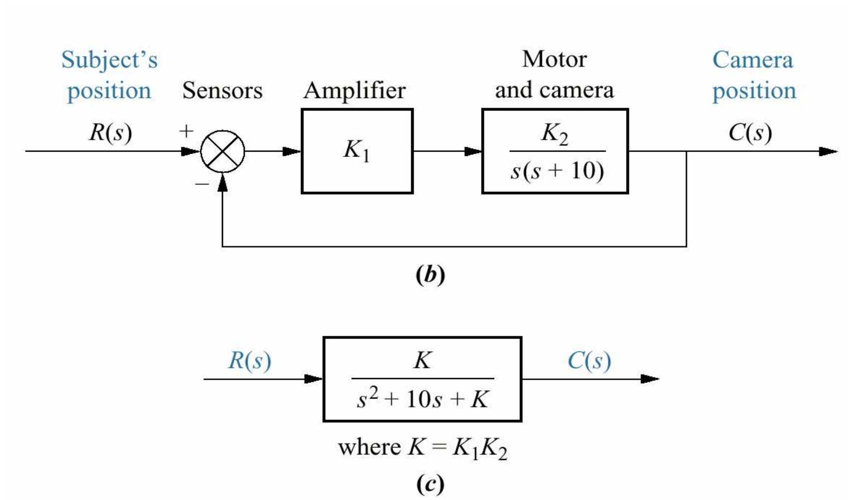
# Lecture 1

# Root Locus

- What is Root-Locus? : A graphical representation of closed loop poles as a system parameter varied.
- Based on Root-Locus graph we can choose the parameter for stability and the desired transient response.

# How does the Root-Locus graph look-like?

– For the system

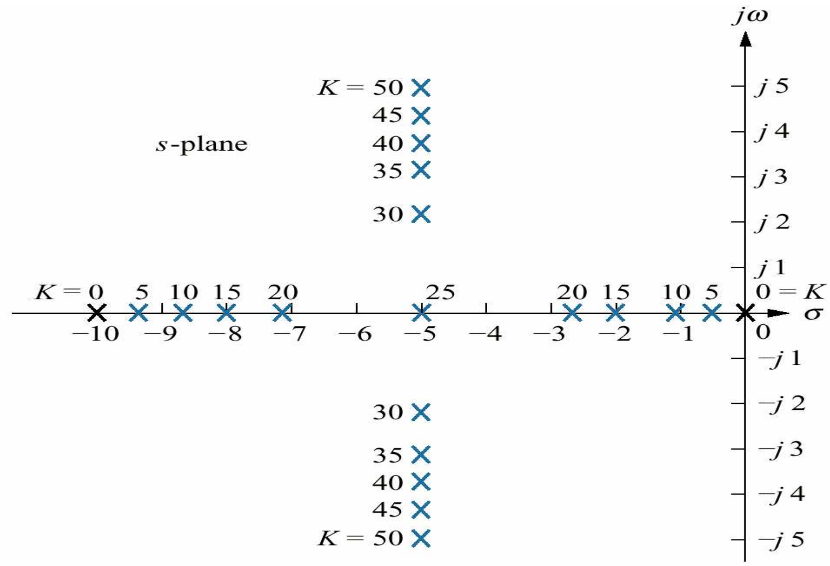



The location of poles as a function of $K$ can be calculated as

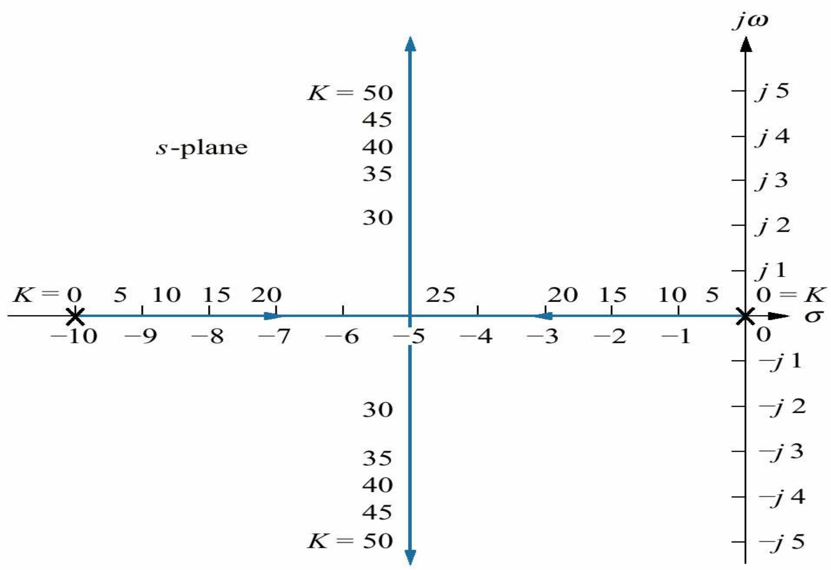
| K | Pole 1 | Pole 2 |
|---|---|---|
| 0 | $-10$ | $0$ |
| 5 | $-9.47$ | $-0.53$ |
| 10 | $-8.87$ | $-1.13$ |
| 15 | $-8.16$ | $-1.84$ |
| 20 | $-7.24$ | $-2.76$ |
| 25 | $-5$ | $-5$ |
| 30 | $-5 + j2.24$ | $-5 - j2.24$ |
| 35 | $-5 + j3.16$ | $-5 - j3.16$ |
| 40 | $-5 + j3.87$ | $-5 - j3.87$ |
| 45 | $-5 + j4.47$ | $-5 - j4.47$ |
| 50 | $-5 + j5$ | $-5 - j5$ |

# Our First Root Locus

The corresponding root locus can be drawn




(a)



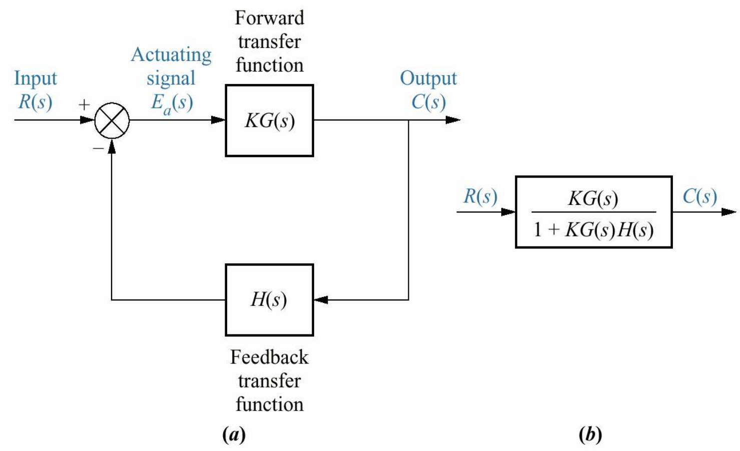

(b)

# Drawing the Root Locus

- How do we draw root locus
  - for more complex systems,
  - and without calculating poles.
- We exploit the properties of Root-Locus to do a rough sketch.
- Therefore, lets explore the properties of root locus.

# Properties of Root Locus

For the closed loop system




- The transfer function is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

- For a given $K$, $s^*$ is a pole if

$$1 + KG(s^*)H(s^*) = 0$$

which is equivalent to

– $|KG(s^*)H(s^*)| = 1$,

– $\angle KG(s^*)H(s^*) = (2m + 1)\pi$

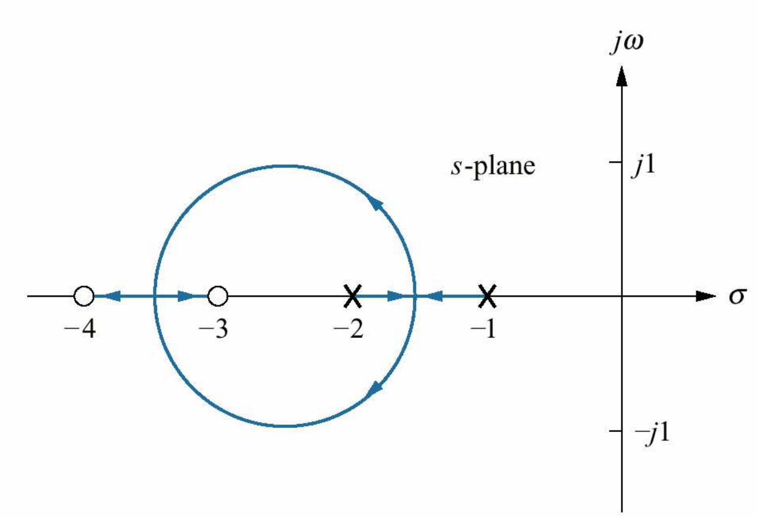
Since $K$ is real and positive these conditions would be equivalent to

– $\angle G(s^*)H(s^*) = (2m + 1)\pi$

– $K = \frac{1}{|G(s^*)H(s^*)|}$

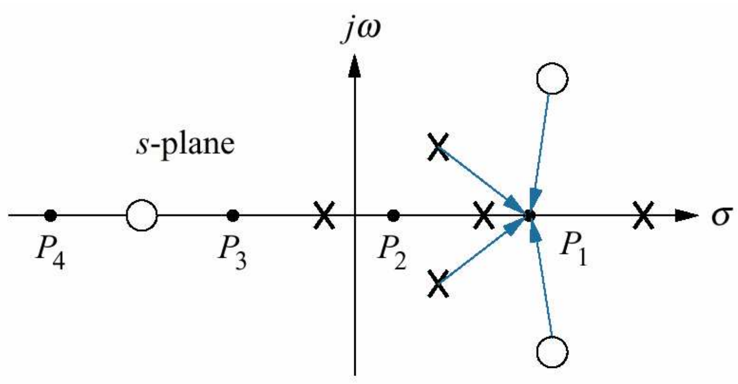
# Number of Branches and Symmetry

- The number of branches of the root locus equals the number of closed loop poles




- Since the poles appear as complex conjugate pairs, root locus is symmetric about real axis

# Real Axis Segments




- Which parts of real line will be a part of root locus?
- Remember the angle condition
$\angle G(\sigma)H(\sigma) = (2m+1)\pi$
- $$\angle G(\sigma)H(\sigma) = \sum \angle(\sigma - z_i) - \sum \angle(\sigma - p_i)$$
- The angle contribution of off-real axis poles and zeros is zero. (Because they appear in complex pairs).
- What matters is the the real axis poles and zeros.
- Rule: If the total number of open loop poles and zeros on the right of a point is odd then that point is part of root-locus.

# Real Axis Segments: Examples

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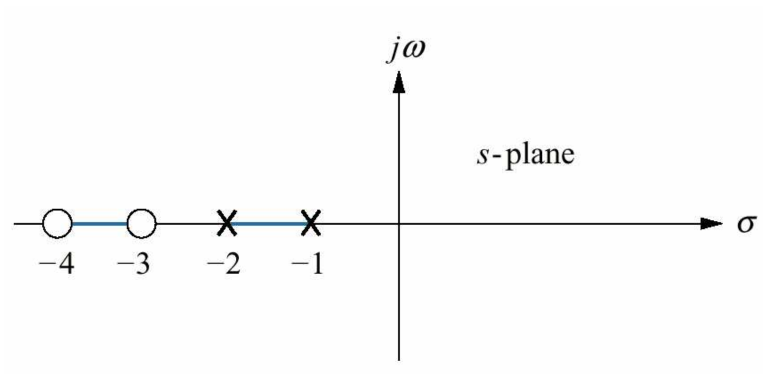
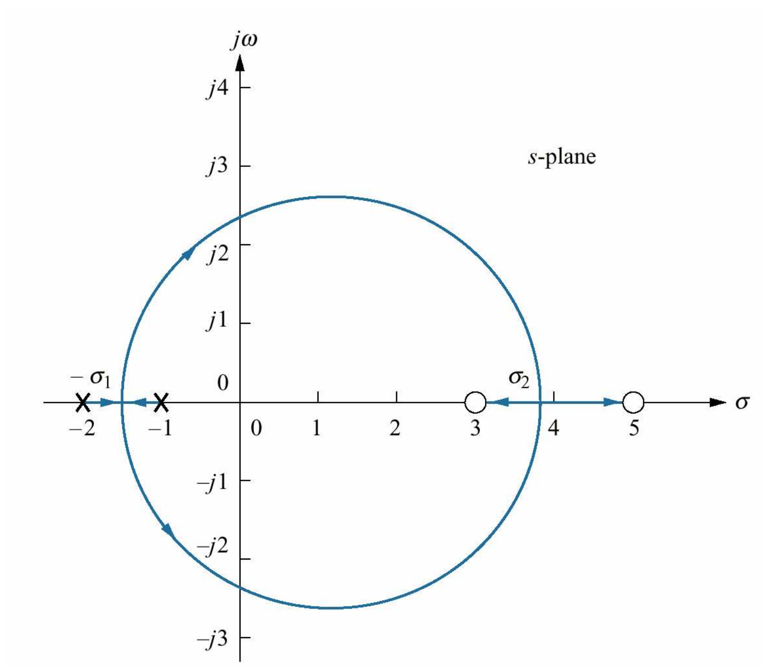
- Example 1




The real axis segments: $[-2,-1]$ and $[-4,-3]$

- Example 2




The real axis segments: $[-2,-1]$ and $[3,5]$

# Start and End Points

- Lets write

$$H(s) = \frac{N_H(s)}{D_H(s)} \qquad G(s) = \frac{N_G(s)}{D_G(s)}$$

Therefore

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)}$$

As a result,

- when $K$ is close to zero

$$T(s) \approx \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s)}$$

i.e. the closed loop poles are essentially the the poles of $G(s)H(s)$.

- when $K$ is large

$$T(s) \approx \frac{KN_G(s)D_H(s)}{KN_G(s)N_H(s)}$$

i.e. the closed loop poles are essentially the zeros of $G(s)H(s)$.

**Conclusion**: *The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.*

# Behavior at Infinity

- What if the number of (finite) open loop poles are more than (finite) open loop zeros, e.g.,

$$KG(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

  – The poles are at $0, -1, -2$

  – The zeros are at $s \to \infty$.

- Let $s$ approach to $\infty$ then

$$KG(s)H(s) \approx \frac{K}{s^3}$$

- Skipping the details, the asymptotes are calculated using formulas:

  – The real axis intercept: The point where the asymptotes merge on the real axis

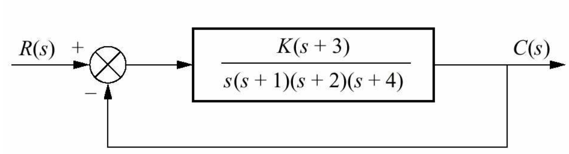
$$\sigma_a = \frac{\Sigma finitepoles - \Sigma finitezeros}{\#finitepoles - \#finitezeros}$$

  – The angles with real line:

$$\theta_a = \frac{(2m+1)\pi}{\#finitepoles - \#finitezeros}$$

# Asymptotes: Example

- Consider the unity feedback system



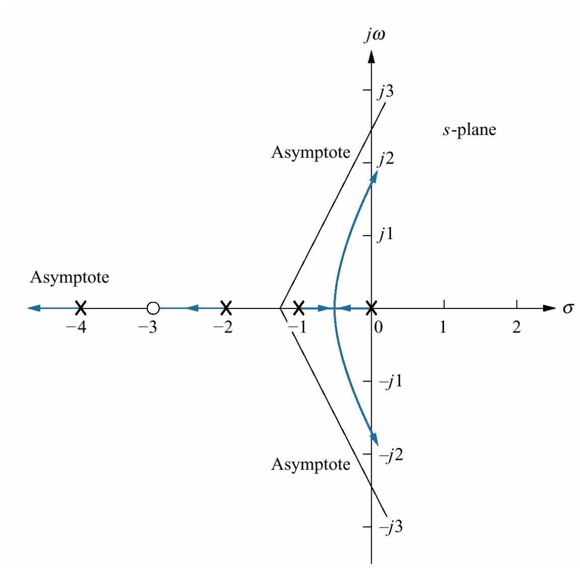

- The real axis intercept for the asymptotes:

$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3}$$

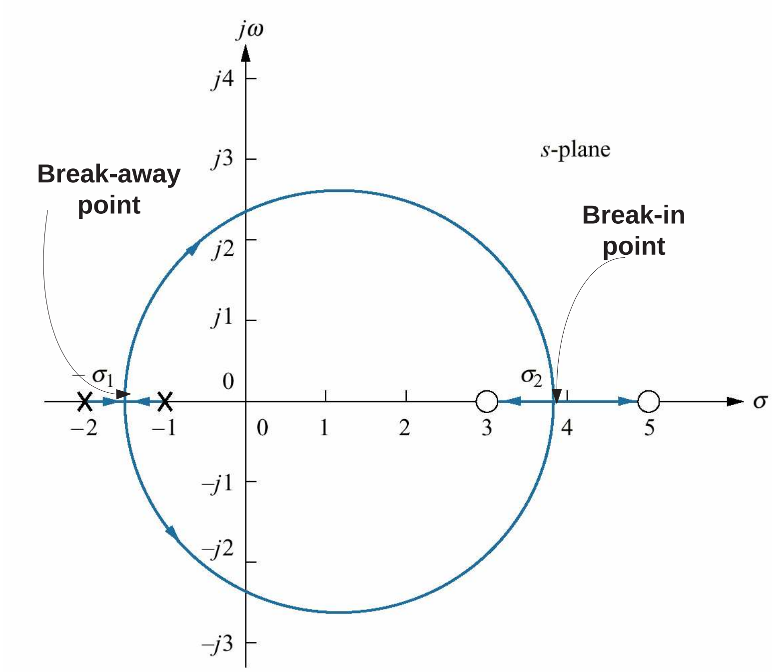
- The angles

$$\theta_a = \frac{(2m+1)\pi}{3}$$

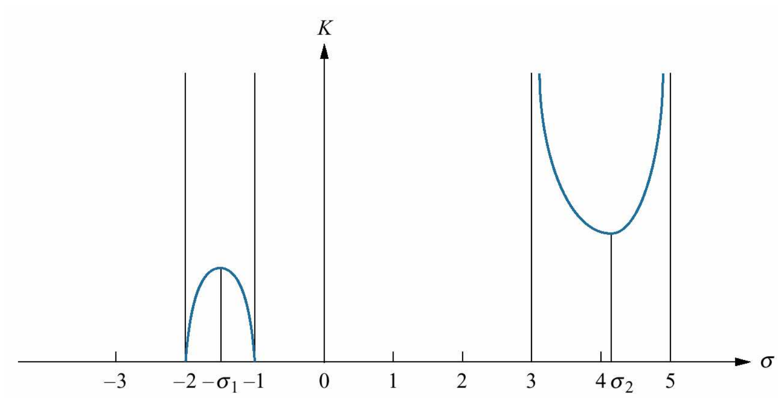
which yields $\frac{\pi}{3}$, $\pi$ and $\frac{5\pi}{3}$



# Break-away and Break-in Points




- Break-away point: The point where root-locus leaves the real axis.
- Break-in point: The point where root locus enters the real axis.
- Variation of $K$ as a function of $\sigma$



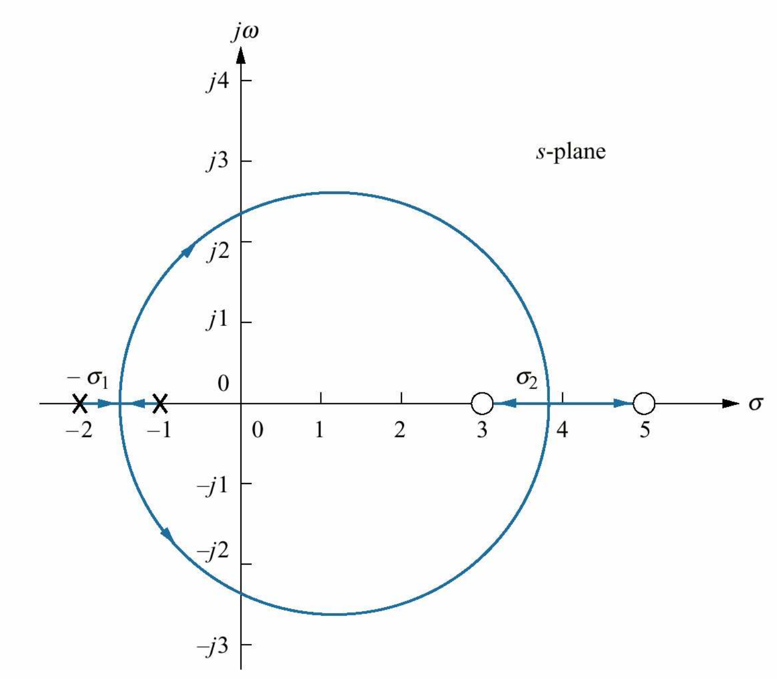

- Note that the curves have their local maximum and minimum points at break-away and break-in points. So the derivative of

$$K = -\frac{1}{G(\sigma)H(\sigma)}$$

should be equal to zero at break-away and break-in points.

# Break-away and Break-in Points: Example

Find the break-away and break-in point of the following figure




**Solution**: From the figure

$$K(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{s^2+3s+1}$$

On the real axis

$$K = -\frac{\sigma^2+3\sigma+2}{\sigma^2-8\sigma+15}$$

Differentiating $K$ with respect to $\sigma$ and equating to zero

$$\frac{dK}{d\sigma} = \frac{11\sigma^2 - 26\sigma - 61}{(\sigma^2 - 8\sigma + 15)^2} = 0$$

which is achieved for $\sigma_1 = -1.45$ and $\sigma_2 = 3.82$.

- it can be shown that a break-away or break-in point satisfy

$$\sum \frac{1}{\sigma + z_i} = \sum \frac{1}{\sigma + p_i}$$

- Applying this formula to our problem, we obtain

$$\frac{1}{\sigma - 3} + \frac{1}{\sigma - 5} = \frac{1}{\sigma + 1} + \frac{1}{\sigma + 2}$$

which would yield

$$11\sigma^2 - 26\sigma - 61 = 0$$

(same as what we obtained before)

# $j\omega$ Axis Crossings

- Use Routh-Hurwitz to find $j\omega$ axis crossings.
- When we have $j\omega$ axis crossings, the Routh-table has all zeros at a row.
- Find the $K$ value for which a row of zeros is achieved in the Routh-table.

**Example:** Consider

$$T(s) = \frac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K}$$

The Routh table

| | | | |
|---|---|---|---|
| $s^4$ | $1$ | $14$ | $3K$ |
| $s^3$ | $7$ | $8+K$ | |
| $s^2$ | $90-K$ | $21K$ | |
| $s^1$ | $\frac{-K^2-65K+720}{90-K}$ | | |
| $s^0$ | $21K$ | | |

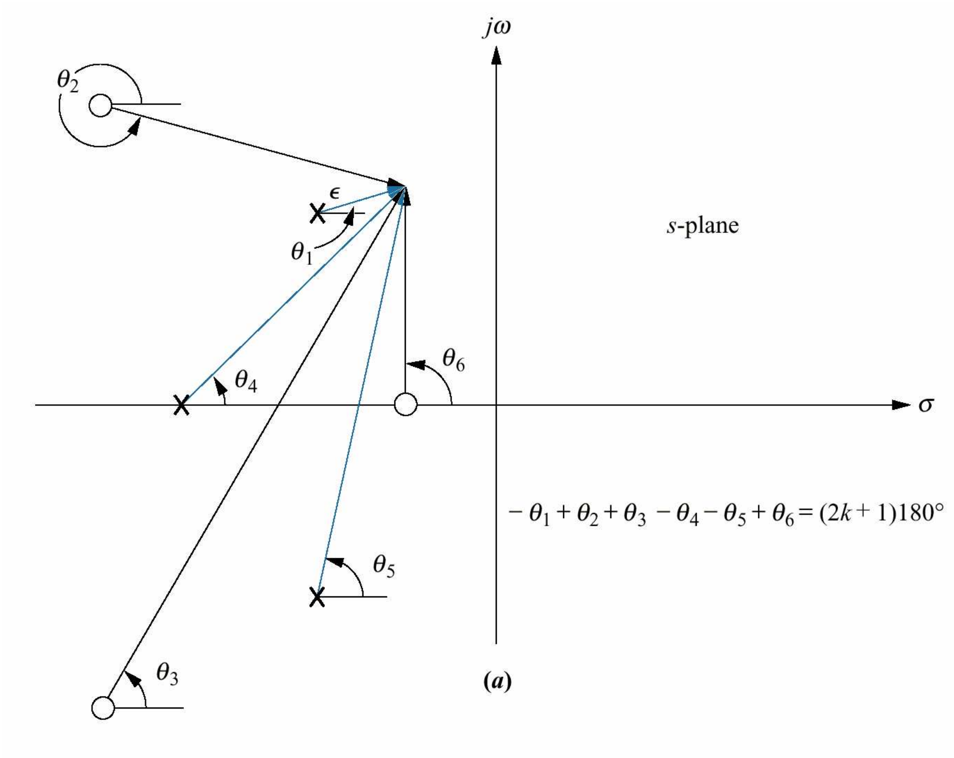
The row $s^1$ is zero for $K = 9.65$. For this $K$, the previous row polynomial is

$$(90-K)s^2 + 21K = 80.35s^2 + 202.7 = 0$$

whose roots are $s = \pm j1.59$.

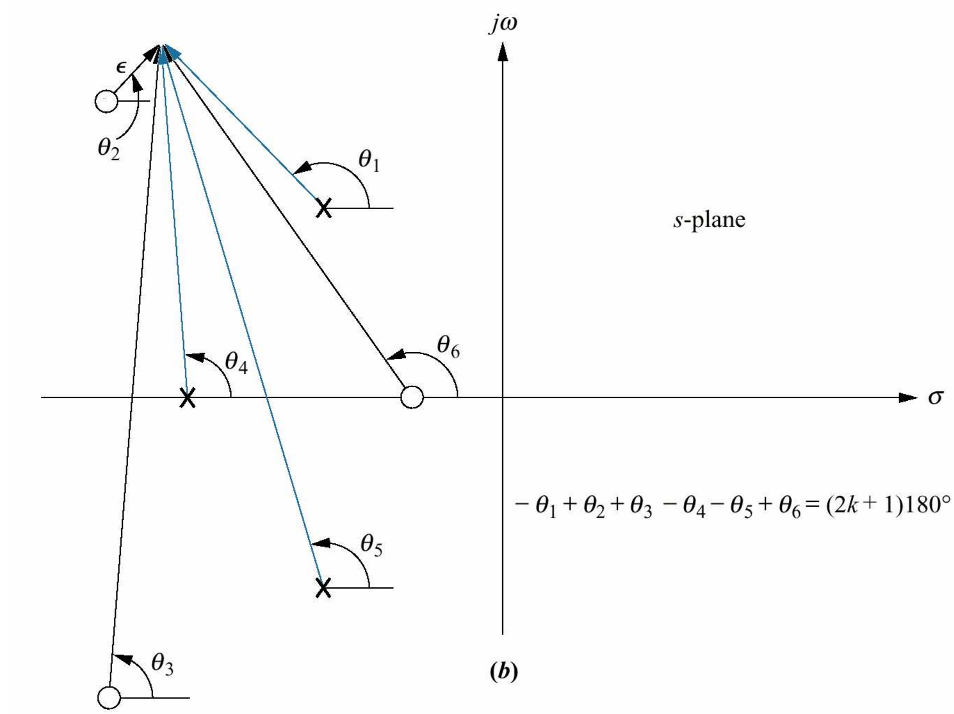
# Angles of Departure and Arrival

- Angles of Departure from Open Loop Poles




$$\theta_1 = \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 - (2k+1)180^o$$

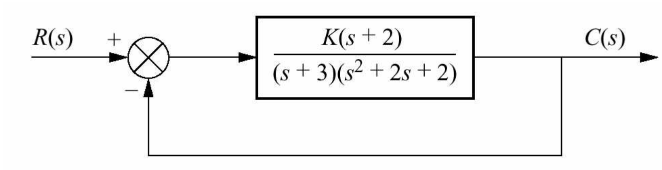
- Angles of Departure form Open Loop Zeros



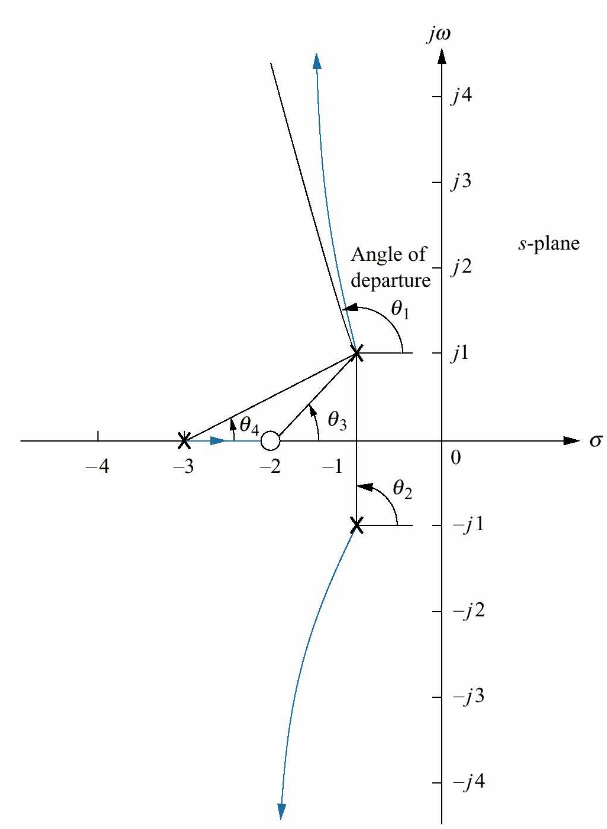

$$\theta_2 = \theta_1 - \theta_3 + \theta_4 + \theta_5 - \theta_6 + (2k+1)180^o$$

# Angle of Departure: Example

Consider the system




The root locus for this system



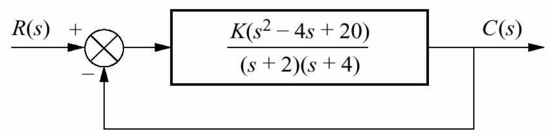

from this figure

$$-\theta_1-\theta_2+\theta_3-\theta_4 = -\theta_1-90^o+\arctan(\frac{1}{1})-\arctan(\frac{1}{2}) = 180^o$$

from which we obtain $\theta_1 = -108.4^o$.

# Root Locus Example

**Problem:** Sketch the Root-Locus of the system



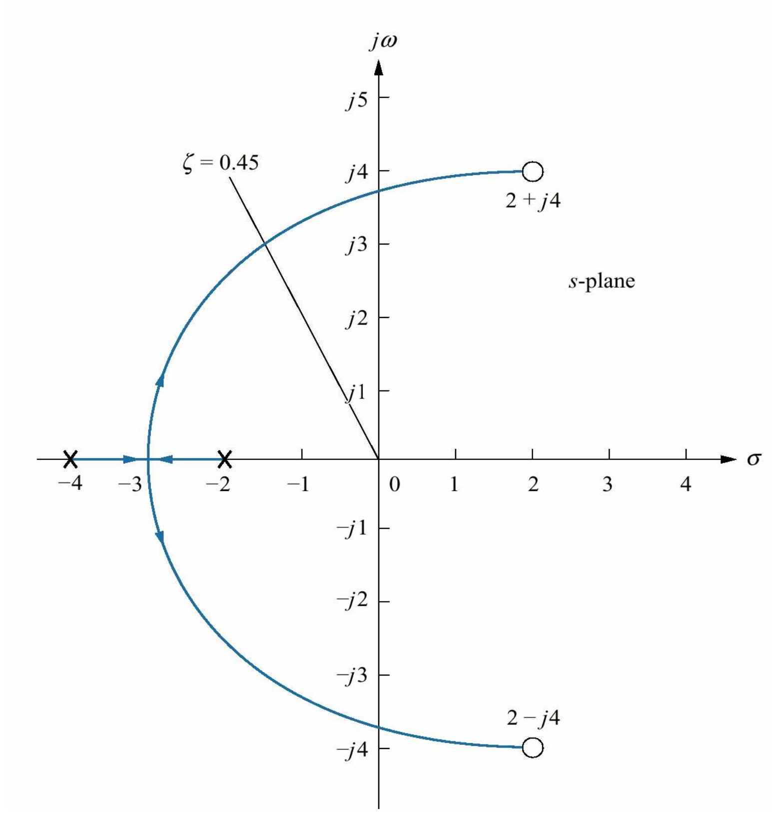

- The number of branches: $2$
- Open Loop Poles: $-2$,$-4$ (starting points)
- Open Loop Zeros: $2 + j4$, $2 - j4$ (ending points)
- Real Axis segments: $[-4, -2]$.
- Number of finite poles $=$ Number of Finite Zeros $\Rightarrow$ No Asymptotes
- Break-away point: Take the derivative of $K = -\frac{1}{\sigma}$

$$\frac{dK}{d\sigma} = -\frac{d}{d\sigma}\frac{(\sigma+2)(\sigma+4)}{\sigma^2 - 4\sigma + 20} = \frac{-10s^2 + 24s + 152}{(\sigma^2 - 4\sigma + 20)^2}$$

  equating to zero we obtain $\sigma_b = -2.87$ and $K = 0.0248$.

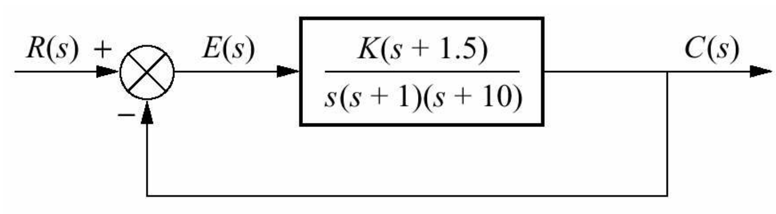
- $j\omega$ axis crossing occurs for $K = 1.5$ and at $\pm j3.9$.
- The root locus crosses $\zeta = 0.45$ line for $K = 0.417$ at $3.4\angle 116.7^o$

# The resulting Root Locus:



# Root Locus Example 2

**Problem:** Sketch the Root-Locus of the third order system




- Number of branches: $3$
- Open Loop Poles: $0, -1 - 10$ (Starting points)
- Open Loop Zero: $-1.5$ (One of the end points)
- Real Axis Segments: $[-1, 0]$ and $[-10 - 1.5]$
- Asymptotes: $\sigma_a = \frac{-11-(-1.5)}{3-1} = -4.75$ and $\theta_a = \frac{\pi}{2}, \frac{3\pi}{2}$.
- Break-in,away points: The derivative of $K = -\frac{1}{G(\sigma)}$ yields

$$\frac{2\sigma^3 + 15.5\sigma^2 + 33\sigma + 15}{()^2}$$

  equating to zero we obtain

  - $\sigma_1 = -0.62$ with gain $K = 2.511$ (Break-away point)
  - $\sigma_2 = -4.4$ with gain $K = 28.89$ (Break-away point)

– $\sigma_3 = -2.8$ with gain $K = 27.91$ (Break-in point)

The resulting root locus