Homework Set #1

Due: Wednesday, October 9, 2013.

1. *A Reminder on Solving Linear Systems of Equations*

   Given $\mathbf{Ax} = \mathbf{b}$, provide solutions for the following $\mathbf{A}, \mathbf{b}$ pairs

   (a)

$$\mathbf{A} = \begin{bmatrix} 2 & 13 & 5 & 6 \\ 0.5 & 3.25 & 5.25 & 2.5 \\ 1.0 & 10.5 & -0.5 & 9 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 16 \\ 7 \\ 3 \end{bmatrix}$$

   (b)

$$\mathbf{A} = \begin{bmatrix} 2 & 13 & 5 & 6 \\ 0.5 & 3.25 & 1.25 & 1.5 \\ 1.0 & 10.5 & -0.5 & 9 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 16 \\ 7 \\ 3 \end{bmatrix}$$

2. *Convolution Operator and Corresponding Matrices...*

   Suppose $\{x_k; k \in \mathbf{Z}\}$ and $\{y_k; k \in \mathbf{Z}\}$ are the input and the corresponding output sequences respectively of a causal linear time invariant system with impulse response $\{h_k; k \in \mathbf{Z}^+\}$, such that,

$$y_k = \sum_{i=0}^{\infty} h_i x_{k-i} \qquad k \in \mathbf{Z}.$$

   (a) Suppose that x_k is a causal sequence, i.e.,

$$x_k = 0 \text{ for } k < 0.$$

   We define the vectors,

$$\mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N) \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N) \end{bmatrix}$$

   Find the matrix, $\mathbf{T}$ such that $\mathbf{y} = \mathbf{Tx}$. The matrix $\mathbf{T}$ describes a mapping from a chunk of input samples to to a chunk of output samples and it is a Toeplitz matrix.

   (b) Suppose that x_k is a sequence, such that

$$x_k = 0 \text{ for } k > 0 \text{and for } k < -N.$$

We define the vectors,

$$\mathbf{x} = \begin{bmatrix} x(0) \\ x(-1) \\ \vdots \\ x(-N) \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N) \end{bmatrix} \tag{1}$$

Find the matrix, $\mathbf{H}$ such that $\mathbf{y} = \mathbf{Hx}$. The matrix $\mathbf{H}$ describes a mapping from a chunk of past input samples to to a chunk of output samples and it is a Hankel matrix.

3. *LU Decomposition Exercise...*

   Find the L, U decomposition for

$$\begin{bmatrix} 6 & 30 & 36 \\ 2 & 10 & 4 \\ 5 & 4 & 2 \end{bmatrix}$$

4. *Yet Another Proof of Cauchy-Schwarz...*

   (a) Suppose $a \geq 0$, $c \geq 0$ and for all $\lambda \in \mathbf{R}$, $a + 2b\lambda + c\lambda^2 \geq 0$. Show that $|b| \leq \sqrt{ac}$.

   (b) Given $\mathbf{u}, \mathbf{w} \in \mathbf{R}$, explain why $(\mathbf{u} + \lambda\mathbf{w})^T(\mathbf{u} + \lambda\mathbf{w}) \geq 0$ for all $\lambda \in \mathbf{R}$.

   (c) Apply a to the quadratic resulting from the expansion of the expression in (b), to get the Cauchy-Schwarz inequality:

$$|v^T w| \leq \sqrt{\mathbf{v}^T\mathbf{v}}\sqrt{\mathbf{w}^T\mathbf{w}}$$

5. *Rank and Matrix Products....*

   For each of the following statements, either show that it is true or give a counterexample

   (a) If $\mathbf{AB}$ is full rank then $\mathbf{A}$ and $\mathbf{B}$ are full rank.

   (b) If $\mathbf{A}$ and $\mathbf{B}$ are full rank then $\mathbf{AB}$ is full rank.

   (c) If $\mathbf{A}$ and $\mathbf{B}$ have zero nullspace, then so does $\mathbf{AB}$.

   (d) If $\mathbf{A}$ and $\mathbf{B}$ are onto, then so is $\mathbf{AB}$.

6. *Fundamental Subspaces and Determinant....*

   Given $\mathbf{A} \in \mathbf{R}^{m \times n}$ show that

   (a) If $\mathcal{R}(\mathbf{A}) = \mathbf{R}^m$, then $det(\mathbf{AA}^T) \neq 0$.

   (b) If $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$ then $det(\mathbf{A}^T\mathbf{A}) \neq 0$.

7. *Characteristic Polynomial of a Square Matrix....*

   Consider the characteristic polynomial $X(s) = det(sI - \mathbf{A})$ of the matrix $\mathbf{A} \in \mathbf{R}^{n \times n}$

(a) Show that $X(s)$ is monic, which means that its leading coefficient is one: $X(s) = s^n + ...$

(b) Show that s^{n-1} coefficient of $X(s)$ is given by $-\textbf{Tr}\,\mathbf{A}$. (Remember $\textbf{Tr}\,\mathbf{A}$ is the trace of the matrix $\mathbf{A}$ which is the sum of the diagonal entries of $\mathbf{A}$, i.e., $\textbf{Tr}\,\mathbf{A} = \sum_{i=1}^{n} A_{ii}$.)

(c) Show that the constant coefficient of $X(s)$ is given by $det(-A)$.

(d) Let $\lambda_1, ..., \lambda_n$ denote the eigenvalues of $\mathbf{A}$, so that

$$X(s) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 = (s - \lambda_1)(s - \lambda_2)...(s - \lambda_n)$$

by equating coefficients show that $a_{n-1} = -\sum_{i=1}^{n} \lambda_i$ and $a_0 = \prod_{i=1}^{n}(-\lambda_i)$. Therefore conclude that $\textbf{Tr}\,\mathbf{A} = \sum_{i=1}^{n} \lambda_i$ and $det\mathbf{A} = \prod_{i=1}^{n} \lambda_i$

8. *Schur's Theorem and Cayley-Hamilton Theorem...*

Schur's Theorem is as follows: Given $\mathbf{A} \in \mathbf{R}^{n\times n}$ with eigenvalues $\lambda_1, ..., \lambda_n$ in any prescribed order, there is a unitary matrix $\mathbf{U}$ such that

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \mathbf{T}$$

is upper triangular with diagonal entries $t_{ii} = \lambda_i$, $i = 1, ..., n$. Remember that a unitary matrix $\mathbf{U}$ has the property $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = I$ where $\mathbf{U}^*$ is the hermitian transpose of $\mathbf{U}$.

(a) (*Cayley-Hamilton Theorem*) Using Schur's Theorem above, prove the Cayley-Hamilton Theorem which states: Let $p_A(t)$ be the characteristic polynomial of $\mathbf{A} \in \mathbf{R}^{n\times n}$, (i.e., $p_{\mathbf{A}}(t) = \det(tI - \mathbf{A})$), then

$$p_{\mathbf{A}}(\mathbf{A}) = 0$$

(b) Using the result of (a) and assuming

$$p_{\mathbf{A}}(t) = t^n + a_{n-1}t^{n-1} + ... + a_1 t + a_0$$

write each of$\mathbf{A}^n, \mathbf{A}^{n+1}, \mathbf{A}^{n+2}$ as polynomials of $\mathbf{A}$ with degree at most $n - 1$. Assuming $\mathbf{A}$ is invertible write $\mathbf{A}^{-1}$ also as a polynomial of $\mathbf{A}$ of degree at most $n - 1$.