ELEC505/ELEC405 Handout #3
Linear Systems Theory Friday, October 11, 2013

Homework Set #2

Due: Friday, October 25, 2013.

1. *Projection Matrices* A matrix $\mathbf{P} \in \mathbf{R}^{n\times n}$ is called an orthogonal projection matrix if $\mathbf{P} = \mathbf{P}^T$ and $\mathbf{P}^2 = \mathbf{P}$.
   (a) Show that if $\mathbf{P}$ is an orthogonal projection matrix then so is $\mathbf{I} - \mathbf{P}$.
   (b) Suppose that the columns of $\mathbf{U} \in \mathbf{R}^{n\times k}$ are orthonormal. Show that $\mathbf{U}\mathbf{U}^T$ is a orthogonal projection matrix.
   (c) Suppose $\mathbf{A} \in \mathbf{R}^{n\times k}$ is full rank with $k \leq n$. Show that $\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ is an orthogonal projection matrix.
   (d) if $S \subseteq \mathbf{R}^n$ and $\mathbf{x} \in \mathbf{R}^n$, the point $\mathbf{y}$ in S closest to x is called the orthogonal projection of $\mathbf{x}$ onto S. Show that if $\mathbf{P}$ is a orthogonal projection matrix, then $\mathbf{y} = \mathbf{P}\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ on $\mathcal{R}(\mathbf{P})$. (Which is why such matrices are called projection matrices...)

2. *Orthogonal matrices*
   (a) Show that if $\mathbf{U}$ and $\mathbf{V}$ are orthogonal then so is $\mathbf{U}\mathbf{V}$.
   (b) Suppose that $\mathbf{U} \in \mathbf{R}^{2\times 2}$ is orthogonal. Show that $\mathbf{U}$ is either a rotation or a reflection. Make clear how you decide whether a given orthogonal $\mathbf{U}$ is a rotation or reflection.

3. Let $\mathbf{A} \in \mathbf{R}^{n\times n}$ and if λ, μ are to distinct eigenvalues of $\mathbf{A}$, i.e. $\mu \neq \lambda$, then show that any left eigenvector of $\mathbf{A}$ corresponding to μ is orthogonal to any right eigenvector of $\mathbf{A}$ corresponding to λ.

4. Suppose $\mathbf{a}, \mathbf{b} \in \mathbf{R}^{n\times n}$ are two given points. Show that the set of points in $\mathbf{R}^n$ that are closer to $\mathbf{a}$ than $\mathbf{b}$ is a half space, i.e.:
   $$\{\mathbf{x} \mid \| \mathbf{x} - \mathbf{a} \| \leq \| \mathbf{x} - \mathbf{b} \|\} = \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} \leq d\}$$
   for appropriate $\mathbf{c} \in \mathbf{R}^n$ and $d \in \mathbf{R}$. Give $\mathbf{c}$ and d explicitly, and draw a picture showing $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and the halfspace (for n=2). Note that the set $\{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} = d\}$ is called an hyperplane.

5. *Reflection through a hyperplane.* Find the matrix $\mathbf{Q} \in \mathbf{R}^{n\times n}$ such that the reflection of $\mathbf{x}$ through the hyperplane $\{\mathbf{z} \mid \mathbf{a}^T\mathbf{z} = 0\}$ is given by $\mathbf{Q}\mathbf{x}$. Verify that matrix $\mathbf{Q}$ is orthogonal. (To reflect $\mathbf{x}$ through the hyperplane means the following: find the point $\mathbf{z}$ on the hyperplane closest to $\mathbf{x}$. Starting from $\mathbf{x}$ go in the direction $\mathbf{z} - \mathbf{x}$ through the hyperplane to a point on the opposite side, which has the same distance to $\mathbf{z}$ as $\mathbf{x}$ does.)

6. (**Bonus**) *Communication Channel*

(a) Suppose a single user is transmitting information signal $\{x_n\}$ to a receiver with a single antenna. $\mathcal{Z}$ transform of the received signal can be written as

$$Y(z) = H(z)X(z)$$

where $H(z)$ corresponds to the stable transfer function corresponding to the channel from the transmitter to the receiver. What is the condition on $H(z)$ such that we can invert it perfectly with a receiver filter $K(z)$ so that we can recover the original transmitted sequence error free. Note that the user is using all of $[-\pi, \pi)$ bandwidth.

(b) Now a single user is transmitting to a receiver using an antenna array with N elements. If we label the $\mathcal{Z}$ transform of the sequence received at antenna i as $Y_i(z)$, then we can write

$$\begin{bmatrix} Y_1(z) \\ Y_2(z) \\ \vdots \\ Y_N(z) \end{bmatrix} = \underbrace{\begin{bmatrix} H_1(z) \\ H_2(z) \\ \vdots \\ H_N(z) \end{bmatrix}}_{\mathbf{H}(z)} X(z).$$

The receiver first filters the signals from different antennas and then adds them to recover the original transmitted sequence, i.e., it forms the signal

$$\hat{X}(z) = K_1(z)Y_1(z) + K_2(z)Y_2(z) + ... + K_N(z)Y_N(z).$$

What is the condition on $\mathbf{H}(z)$ for the error free recovery of the transmitted sequence $\{x_n\}$. Note that the user is using all of $[-\pi, \pi)$ bandwidth.

(c) In a typical wireless reverse link scenario, multiple users transmit to the same base station. Suppose there are M users transmitting to a base station and the base station employs an antenna array with N elements. If we label the $\mathcal{Z}$ transform of the sequence received at antenna i as $Y_i(z)$, then we can write

$$\begin{bmatrix} Y_1(z) \\ Y_2(z) \\ \vdots \\ Y_N(z) \end{bmatrix} = \underbrace{\begin{bmatrix} H_{11}(z) & H_{12}(z) & ... & H_{1M}(z) \\ H_{21}(z) & H_{22}(z) & ... & H_{2M}(z) \\ \vdots & & & \\ H_{N1}(z) & H_{22}(z) & ... & H_{NM}(z) \end{bmatrix}}_{\mathbf{H}(z)} \begin{bmatrix} X_1(z) \\ X_2(z) \\ \vdots \\ X_M(z) \end{bmatrix}.$$

For each user, the receiver first filters the signals from different antennas and then adds them to recover the original transmitted sequence, i.e., for user i it forms the signal

$$\hat{X}_i(z) = K_{i1}(z)Y_1(z) + K_{i2}(z)Y_2(z) + ...K_{iN}(z)Y_N(z)$$

for $i = 1, ..., M$. What is the condition on $\mathbf{H}(z)$ for the error free recovery of all users. Note that each user is using all of $[-\pi, \pi)$ bandwidth.