ELEC505/ELEC405 Handout #4
Linear Systems Theory Friday, November 1, 2013

Homework Set #3

Due: Monday, November 11, 2013.

1. *DFT as an Orthogonal Basis Change*

Consider $\mathcal{C}^N$ the vector space of N dimensional complex vectors. We can define a basis $\mathcal{F}$=$\{\mathbf{f}_1, \ldots, \mathbf{f}_N\}$ where

$$\mathbf{f}_k = \begin{bmatrix} f_{k,1} \\ f_{k,2} \\ \vdots \\ f_{k,N} \end{bmatrix}, \quad f_{k,l} = \frac{1}{N} e^{\frac{j2\pi(k-1)(l-1)}{N}}. \tag{1}$$

(a) Is $\mathcal{F}$ an orthogonal basis? Is it an orthonormal basis?

(b) Define the matrix

$$\mathbf{F} = \begin{bmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \ldots & \mathbf{f}_N \end{bmatrix}. \tag{2}$$

Is $\mathbf{F}$ unitary? What is the inverse of $\mathbf{F}$?

(c) Write down $\mathbf{F}$ for $N = 4$. What are the coordinates of $\mathbf{x} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$ corresponding to the basis $\mathcal{F}$? What is the FFT of $\mathbf{x}$?

(d) Suppose $\{x_n : n = 0, \ldots 3\}, \{h_n : n = 0, \ldots 3\}$ and $\{y_n : n = 0, \ldots 3\}$ are discrete time sequences of length 4. We are also given that y_n is equal to the circular convolution of x_n and h_n, i.e.,

$$y_n = h_n \circledast x_n$$

Defining

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

we would like to write the relation

$$\mathbf{y} = \mathbf{H}\mathbf{x} \tag{3}$$

Find $\mathbf{H}$ in terms of h_n, $n = 0, \ldots 3$.

(e) Check out the following multiplication:

$$\mathbf{H}\mathbf{f}_k \tag{4}$$

What is your comment about the result?

(f) Pick arbitrary values for $\{h_0, h_1, h_2, h_3\}$. Find the eigenvalues and eigenvectors of $\mathbf{H}$ using MATLAB, by typing

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[V D]=eig(H)
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where $\mathbf{V}$ is the matrix containing eigenvectors, and $\mathbf{D}$ is the matrix containing eigenvalues. Compare $\mathbf{V}$ with $\mathbf{F}$, what is your comment.

(g) If we change the basis from the standard basis to $\mathcal{F}$, what would be the relation between $\breve{\mathbf{y}}$ and $\breve{\mathbf{x}}$, where $\breve{\mathbf{y}}$ and $\breve{\mathbf{x}}$ are the new coordinate vectors for $\mathbf{y}$ and $\mathbf{x}$ in Eq. (3) respectively, corresponding to basis $\mathcal{F}$. In other words find $\breve{\mathbf{H}}$ where $\breve{\mathbf{y}} = \breve{\mathbf{H}}\breve{\mathbf{x}}$. What is the effect of basis change as far as the mapping in (3) is concerned? Note that this comment is independent of the values of $\{h_0, h_1, \ldots, h_{N-1}\}$....

2. *Complex vs. Real*

Let

- $V_1 = \mathcal{C}^n$, i.e., the n-dimensional complex vector space, equipped with the inner product
$$< \mathbf{x}, \mathbf{y} >_C = \mathbf{y}^* \mathbf{x}, \tag{5}$$
- $V_2 = \Re^{2n}$, i.e., the $2n$-dimensional real vector space, equipped with the inner product
$$< \mathbf{u}, \mathbf{v} >_R = \mathbf{v}^T \mathbf{u}, \tag{6}$$
- $\Gamma : \mathcal{C}^n \to \Re^{2n}$ is a mapping between the two vector spaces, where
$$\Gamma(\mathbf{x}) = \begin{bmatrix} \Re e\{\mathbf{x}\} \\ \mathcal{I}m\{\mathbf{x}\} \end{bmatrix}, \tag{7}$$
- $\mathbf{A} \in \mathcal{C}^{n \times n}$ define a linear mapping $f : \mathcal{C}^n \to \mathcal{C}^n$, such that for any $\mathbf{x} \in \mathcal{C}^n$
$$f(\mathbf{x}) = \mathbf{A}\mathbf{x}, \tag{8}$$
- $g : \Re^{2n} \to \Re^{2n}$ is the mapping corresponding to f such that, for any $\mathbf{x} \in V_1$
$$g(\Gamma(\mathbf{x})) = \Gamma(f(\mathbf{x})). \tag{9}$$

If we explicitly write the linear mapping $\mathbf{g}$ as
$$g(\mathbf{u}) = \mathcal{A}\mathbf{u}, \qquad \forall \mathbf{u} \in \Re^{2n}, \tag{10}$$

(a) Write $\mathcal{A}$ in terms of $\mathbf{A}$.

(b) What type of matrix is $\mathcal{A}$, if $\mathbf{A}$ is Hermitian?

(c) What type of matrix is $\mathcal{A}$, if $\mathbf{A}$ is Unitary?

(d) What type of matrix is $\mathcal{A}$, if $\mathbf{A}$ is Skew-Hermitian?

(e) For $\mathbf{x}, \mathbf{y} \in C^n$, write $< \Gamma(\mathbf{x}), \Gamma(\mathbf{y}) >_R$, in terms of $< ., . >_C$.

3. *Polar Form*

Show that each nonsingular matrix $A \in \mathbf{C}^{n \times n}$ can be factored as $A = RU$, where R is hermitian positive definite and U is unitary. (This is the matrix analog of the polar form of a complex number $z = re^{j\Theta}$.) Hint: Consider the use of the matrix AA^*.