ELEC505/ELEC405 Handout #11
Linear Systems Theory Tuesday, December 3, 2013

Homework Set #5

Due: Wednesday, December 11, 2013.

1. *The Positive Semidefinite Cone*

   Definition (Convex Cone): Given a vector space $\mathcal{V}$, the set $\mathcal{M} \subset \mathcal{V}$ is a convex cone if and only if for every $A, B \in \mathcal{M}$ and $\theta_1 \geq 0, \theta_2 \geq 0$, $\theta_1 A + \theta_2 B \in \mathcal{M}$.

   (a) Show that the set of $n \times n$ real symmetric positive semidefinite matrices, denoted by $\mathbf{S}_+^n$ is a convex cone.

   (b) Now consider the special case $\mathbf{S}_+^2$. Derive the conditions for the symmetric matrix

   $$\mathbf{X} = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \tag{1}$$

   to be a positive semi-definite matrix. Based on the derived conditions, draw the boundary of the corresponding convex cone in the xyz plane using MATLAB.

2. *QR Factorization: Orthogonal Triangularization through Rotations*

   Consider the 2×2 orthogonal matrix, which corresponds to rotation in R^2.

   $$Q = \begin{bmatrix} \cos(\Theta) & \sin(\Theta) \\ -\sin(\Theta) & \cos(\Theta) \end{bmatrix}$$

   (a) Show how you can use Q to obtain a upper-triangular matrix from a 2×2 matrix

   $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

   Based on this result, write a QR factorization of A.

   (b) Now show that following matrices are unitary:

   $$J_1 = \begin{bmatrix} \cos(\Theta) & \sin(\Theta) & 0 \\ -\sin(\Theta) & \cos(\Theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, J_2 = \begin{bmatrix} \cos(\Theta) & 0 & \sin(\Theta) \\ 0 & 1 & 0 \\ -\sin(\Theta) & 0 & \cos(\Theta) \end{bmatrix}, J_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\Theta) & \sin(\Theta) \\ 0 & -\sin(\Theta) & \cos(\Theta) \end{bmatrix}$$

   Explain which geometrical operations these matrices correspond to, highlighting their differences.

   (c) Using the results of the parts (a) and (b), propose a procedure to obtain the QR factorization of a 3×3 matrices where you use orthogonal triangularization with orthogonal operators correspond to rotation.

   (d) How do you generalize your procedure in (c) to general $m \times m$ matrices

3. *MIL*

   Find a simplified expression for the inverse of the following $n \times n$ matrix:

$$\mathbf{I} + \mathbf{x}\mathbf{y}^* \tag{2}$$

   where $\mathbf{x}, \mathbf{y}$ are $n \times 1$ vectors. What is the required condition for invertibility?

4. *Matrix Norms*

   Show that if $\|\mathbf{A}\|_2 < 1$, $\mathbf{I} - \mathbf{A}$ is invertible.

5. *Matrix Norms and Projection Matrices*

   Let $\mathbf{P}$ be an arbitrary orthogonal projection matrix, i.e., $\mathbf{P}^2 = \mathbf{P}$ and $\mathbf{P}^* = \mathbf{P}$. What can you say about $\|\mathbf{P}\|_2$?