

**Homework Set #6**

Due: Wednesday, December 18, 2013.

1. *Image of 2-norm Ball...*

Let  $S_n$  denote unit 2-norm ball in  $\mathfrak{R}^n$ . Therefore, we can write  $S_n = \{\mathbf{x} \in \mathfrak{R}^n \mid \|\mathbf{x}\|_2 = 1\}$ . Let  $\mathbf{A} \in \mathfrak{R}^{m \times n}$  be a tall matrix (i.e.,  $m > n$ ), which is not necessarily full rank. Show that the image of  $S_n$  under the mapping

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x} \tag{1}$$

is an hyperellipse.

2. *A Useful Fact About the Spectral Norm*

For a given matrix  $\mathbf{A}$ , show that  $\|\mathbf{A}\|_2 < 1$  if and only if the matrix

$$\begin{bmatrix} \mathbf{I} & \mathbf{A}^* \\ \mathbf{A} & \mathbf{I} \end{bmatrix} \tag{2}$$

is positive.

3. *SVD muscle exercise (Trefethen & Bau) 4.1*

Determine the SVDs of the following matrices by hand calculation:

$$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{3}$$

4. *Transpose out of this world (Trefethen & Bau) 4.2*

Suppose  $\mathbf{A}$  is an  $m \times n$  matrix and  $\mathbf{B}$  is the  $n \times m$  matrix obtained by rotating  $\mathbf{A}$  ninety degrees clockwise on paper (not exactly a standard mathematical transformation!). Do  $\mathbf{A}$  and  $\mathbf{B}$  have same singular values? prove that answer is yes or give a counterexample.

5. *An encounter of EVD vs. SVD (Trefethen & Bau) 5.4*

Suppose  $\mathbf{A} \in \mathcal{C}^{m \times m}$  has an SVD  $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$ . Find an eigenvalue decomposition of the  $2m \times 2m$  matrix

$$\begin{bmatrix} \mathbf{0} & \mathbf{A}^* \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \tag{4}$$

6. *Polynomial Approximation*

Obtain a 4th order polynomial approximation  $p(x)$  for the function  $f(x) = e^{-\cos(2\pi x)}$  where  $x \in [0, 1]$  such that

$$\int_0^1 |f(x) - p(x)|^2 dx \tag{5}$$

is minimized. Plot the original function and its polynomial approximation on the same figure window.