Homework Set #3

Due: Tuesday, March 11, 2014.

Please attach the MATLAB codes for the problems with the MATLAB parts.

1. *Bias and Variance..... Kay 1.4*

   It is desired to estimate the value of a DC level A in WGN based on the observations

$$x[n] = A + w[n] \qquad n = 0, 1, \ldots, N-1, \tag{1}$$

   where $w[n]$ is zero mean and uncorrelated , and each sample has variance $\sigma^2 = 1$. Consider the two estimators

$$\hat{A}_1 = \frac{1}{N}\sum_{n=0}^{N-1} x[n], \tag{2}$$

$$\hat{A}_2 = \frac{1}{N+2}(2x[0] + \sum_{n=1}^{N-2} x[n] + 2x[N-1]). \tag{3}$$

   Which one is better? (consider bias and variance as the comparison criteria). Does it depend on the value of A?

2. *Estimating Variance.... Kay 2.1*

   The data $\{x[0], x[1], \ldots, x[N-1]\}$ are observed where the $x[n]'s$ are independent and identically distributed (IID) as $\mathcal{N}(0, \sigma^2)$. We wish to estimate the variance σ^2 as

$$\hat{\sigma^2} = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]. \tag{4}$$

   Is this an unbiased estimator? Find the variance of $\hat{\sigma^2}$ and examine what happens as $N \to \infty$.

3. *Bias-Variance Trade-off*

   Consider the DC signal estimation problem in Problem 1. Suppose A is known to be magnitude bounded by A_0,i.e.,

$$|A| \leq A_0. \tag{5}$$

   Can you recommend another estimator whose Mean Square Error performance is better than $\hat{A}_1$ in Problem 1.

4. *Unbiased Estimator .... Kay 2.2*

   Consider the data $\{x[0], x[1], \ldots, x[N-1]\}$, where each sample is distributed as $\mathcal{U}[0, \Theta]$, where $0 < \Theta < \infty$, and the samples are IID. Can you find an unbiased estimator for Θ?

5. *Simple CRLB Exercise ... Kay 3.2*

If a single sample is observed as

$$x[0] = A + w[0], \tag{6}$$

where $w[0]$ has the arbitrary pdf p_w and we want to estimate A.

(a) Derive the CRLB expression for the estimator $\hat{A}$ for a general p_w.

(b) Refine your CRLB expression for the case when $w[0]$ is Laplacian, i.e.,

$$p_w(u) = \frac{1}{\sqrt{2}\sigma} e^{-\frac{\sqrt{2}|u|}{\sigma}}. \tag{7}$$

6. *Score Vector for Poisson Distribution*

Let the observations $\{x[0], x[1], \ldots, x[N-1]\}$ be IID with the Poisson distribution given by

$$p_{x[i]}(k) = e^{-g(\mathbf{\Theta})} \frac{g(\mathbf{\Theta})^k}{k!} \tag{8}$$

where $g(\mathbf{\Theta})$ is a smooth function of the parameter vector $\mathbf{\Theta}$.

(a) Find the score vector $\mathbf{s}_{\mathbf{\Theta}} = \nabla_{\mathbf{\Theta}} \ln(p_{\mathbf{x}}(\mathbf{x}; \mathbf{\Theta}))$, where $\mathbf{x} = \begin{bmatrix} x[0] & x[1] & \ldots & x[N-1] \end{bmatrix}^T$ is the observation vector.

(b) Find $E(\mathbf{s}_{\mathbf{\Theta}})$.

(c) For $g(\Theta) = \Theta^2$, $\Theta \in \Re$, find the variance of s_Θ.

(d) Find the distribution of the score function in part (c).

(e) MATLAB part: For the example in part (c), and for $\Theta = 2$ and $N = 10$,

- generate 1000 independent realizations of the $\mathbf{x}$ vector.
- find corresponding s samples.
- Obtain an estimate of the distribution of the s_Θ using *hist* function with a fixed bin $[-40 : 40]$. You need to normalize the vector obtained from *hist* function such that the sum of its elements is equal to 1. Plot this distribution using *stem* function.
- On the same figure, plot the true distribution of s_Θ, you found in part (d). Use a different marker (e.g. 'r*') for the *stem* function so that two figures can be differentiated.

(f) What is the CRLB for Θ for the scenario in part (e).