ECOE530/ELEC430 Handout #4
Detection and Estimation Theory Tuesday, March 11, 2014

Homework Set #4
Due: Wednesday, March 19, 2014.

1. *Score Vector for Poisson Distribution*

   Let the observations $\{x[0], x[1], \ldots, x[N-1]\}$ be IID with the Poisson distribution given by

$$p_{x[i]}(k) = e^{-g(\boldsymbol{\Theta})} \frac{g(\boldsymbol{\Theta})^k}{k!} \tag{1}$$

   where $g(\boldsymbol{\Theta})$ is a smooth function of the parameter vector $\boldsymbol{\Theta}$.

   (a) Find the score vector $\mathbf{s}_{\boldsymbol{\Theta}} = \nabla_{\boldsymbol{\Theta}} \ln(p_{\mathbf{x}}(\mathbf{x}; \boldsymbol{\Theta}))$, where $\mathbf{x} = \left[\begin{array}{cccc} x[0] & x[1] & \ldots & x[N-1] \end{array}\right]^T$ is the observation vector.

   (b) Find $E(\mathbf{s}_{\boldsymbol{\Theta}})$.

   (c) For $g(\Theta) = \Theta^2$, $\Theta \in \Re$, find the variance of s_Θ.

   (d) Find the distribution of the score function in part (c).

   (e) MATLAB part: For the example in part (c), and for $\Theta = 2$ and $N = 10$,
      - generate 1000 independent realizations of the $\mathbf{x}$ vector.
      - find corresponding s samples.
      - Obtain an estimate of the distribution of the s_Θ using *hist* function with a fixed bin $[-40 : 40]$. You need to normalize the vector obtained from *hist* function such that the sum of its elements is equal to 1. Plot this distribution using *stem* function.
      - On the same figure, plot the true distribution of s_Θ, you found in part (d). Use a different marker (e.g. 'r*') for the *stem* function so that two figures can be differentiated.

   (f) What is the CRLB for Θ for the scenario in part (e).

2. *CRLB and Efficient Estimator .... Kay 3.3*

   The data

$$x[n] = Ar^n + w[n], \qquad , n = 0, 1, \ldots, N-1 \tag{2}$$

   are observed, where $w[n]$ is WGN with variance σ^2 and $r > 0$ is known. Find the CRLB for A. Show that an efficient estimator exists and find its variance. What happens tovariance as $N \to \infty$ for various values of r?

3. *Polynomial Fitting Problem .... Kay 3.13*

   As a generalization of the line fitting problem, consider the polynomial or curve fitting problem. The data model is

$$x[n] = \sum_{k=0}^{p-1} A_k n^k + w[n], \tag{3}$$

for $n = 0, 1, \ldots, N-1$ As before, $w[n]$ is WGN with variance σ^2. It is desired to estimate $\{A_0, A_1, \ldots, A_{p-1}\}$. Find the Fisher information matrix for this problem.

4. *CRLB and Correlated Noise .... Kay 3.9*

   We observe two samples of a DC level in a correlated Gaussian noise

$$\begin{aligned} x[0] &= A + w[0] & (4) \\ x[1] &= A + w[1] & (5) \end{aligned}$$

   where $\mathbf{w} = \begin{bmatrix} w[0] & w[1] \end{bmatrix}^T$ is zero mean with covariance matrix

$$\mathbf{C} = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \tag{6}$$

   The parameter ρ is the correlation coefficient between $w[0]$ and $w[1]$. Compute the CRLB of A and compare it to the case when $w[n]$ is WGN or $\rho = 0$. Also, explain what happens when $\rho \to \pm 1$ (please do not just provide the limiting CRLB value discuss the implication of this). Finally, comment on the additivity of the Fisher information for nonindependent observations.