ECOE530/ELEC430 Handout #4
Detection and Estimation Theory Thursday, March 20, 2014

Homework Set #5

Due: Monday, March 31, 2014.

1. *A "Can't Skip" Matrix Calculus Exercise*

   Find $\frac{\partial ln(det(\mathbf{I}+\mathbf{A}\mathbf{X}^{-1}\mathbf{B}))}{\partial \mathbf{X}}$.

2. *Bivariate Gaussian: Unknown Correlation Coefficient.... Kay 3.14*

   Independent bivariate Gaussian samples $\mathbf{x}[0], \mathbf{x}[1], \ldots, \mathbf{x}[N-1]\}$ are observed. Each observation is a 2×1 vector which is distributed as $\mathbf{x}[n] \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$ and

   $$\mathbf{C} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

   Find the CRLB for the correlation coefficient ρ.

3. *Exponential Family....*

   Exponential family constitutes an important parametric family of distributions, where the pdf of an exponentially distributed vector $\mathbf{x}$ can be written as

   $$p(\mathbf{x}; \boldsymbol{\Theta}) = \alpha(\boldsymbol{\Theta})\beta(\mathbf{x})e^{\gamma(\boldsymbol{\Theta})^T \eta(\mathbf{x})}, \tag{1}$$

   where

   - $\alpha(\boldsymbol{\Theta})$ is a scalar function of $\boldsymbol{\Theta}$ only,
   - $\beta(\mathbf{x})$ is a scalar function of $\mathbf{x}$ only,
   - $\gamma(\boldsymbol{\Theta}) = \begin{bmatrix} \gamma_1(\boldsymbol{\Theta}) & \ldots & \gamma_L(\boldsymbol{\Theta}) \end{bmatrix}^T$ is a vector function of $\boldsymbol{\Theta}$ only,
   - $\eta(\mathbf{x}) = \begin{bmatrix} \eta_1(\mathbf{x}) & \ldots & \eta_L(\mathbf{x}) \end{bmatrix}^T$ is a vector function of $\mathbf{x}$ only.

   (a) Show that the parametric Gaussian family $\mathcal{N}(\mu(\boldsymbol{\Theta}), \boldsymbol{\Sigma})$ is a special case of exponential family, by finding $\alpha, \beta, \gamma, \eta$ functions for the corresponding pdf.

   (b) Show that the Poisson distribution whose pdf can be written as

   $$p(\mathbf{x}; \boldsymbol{\Theta}) = e^{-\sum_{k=1}^{n} g_k(\boldsymbol{\Theta})} \frac{e^{\sum_{k=1}^{n} x_k ln(g_k(\boldsymbol{\Theta}))}}{\prod_{k=1}^{n} x_k!} \tag{2}$$

   by finding the corresponding $\alpha, \beta, \gamma, \eta$ functions.

   (c) Using the fact that the expression in Eq, (1) is a pdf, show that $\alpha(\boldsymbol{\Theta})$ is completely determined by $\gamma(\boldsymbol{\Theta})$.

   (d) Find the expression for the score vector corresponding to exponential family.

(e) As the exponential family satisfies the regularity condition, find

$$\mu_{\eta(\mathbf{x})} = E(\eta(\mathbf{x}))$$

in terms of exponential pdf parameters.

(f) Combine the results of previous parts to write the score vector in terms of $\gamma(\boldsymbol{\Theta})$, $\eta(\mathbf{x})$ and $\mu_{\eta(\mathbf{x})}$.

(g) Find the Fisher information matrix for the exponential family in terms of covariance matrix of $\eta(\mathbf{x})$.

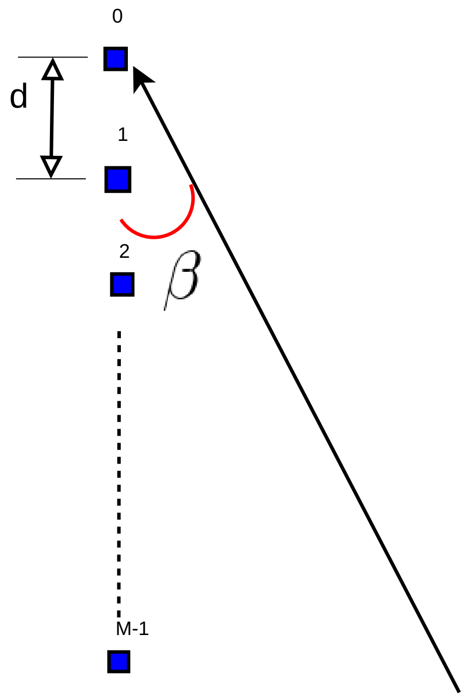

Figure 1: Direction Finding Problem Setup for Problem 4

4. *Direction Finding and CRLB....*

You are working in a military electronics company called MIELIND and your task is to design an antenna array to figure out the direction of an incoming electromagnetic wave. The design consists of a uniform linear array with M antennas (Antenna-0, to Antenna-(M-1)) as shown in Figure 1. Antenna separation is specified with the d parameter. The angle of the electromagnetic wave is defined as β, which is the angle between the array line and the electromagnetic wave direction line. We record a single snapshot of both in phase and out of phase components of the measured signal value at each antenna which are specified by x_n^I and x_n^Q respectively for $n = 0, \ldots, M-1$. Here

$$x_n^I = s_n^I + w_n^I \quad (3)$$
$$x_n^Q = s_n^Q + w_n^Q \quad (4)$$

for $n = 0, \ldots, M-1$, where

- $s_n^I = Acos(2\pi F_o \frac{d}{c} cos(\beta)n + \phi)$,
- $s_n^Q = Asin(2\pi F_o \frac{d}{c} cos(\beta)n + \phi)$,
- F_o is the frequency of the electromagnetic wave
- A is the unknown amplitude of the electromagnetic wave
- ϕ is the unknown phase (at Antenna-0)
- $w_n^I \sim \mathcal{N}(0, \sigma^2)$ is the in-phase component noise at Antenna-n,
- $w_n^Q \sim \mathcal{N}(0, \sigma^2)$ is the quadrature-phase component noise at Antenna-n,
- σ^2 is assumed to be known,
- in-phase and quadrature-phase components of the noise are independent,
- noises at different antennas are independent,
- $d = \frac{\lambda}{2}$ where $\lambda = \frac{c}{F_o}$.

You are to explore the precision limits of the antenna array system in resolving the angle of arrival:

(a) For the unknown parameters $\Theta = \begin{bmatrix} A & \beta & \phi \end{bmatrix}^T$, find the Fisher Information Matrix, in terms of SNR $\eta = \frac{A^2}{\sigma^2}$ and other parameters.

(b) Find the CRLB for the unbiased estimate of Θ.

For the following parts, draw the corresponding curves on the same graph:

(c) For 0 dB SNR and 5 antennas, draw the lower bound on the standard deviation of error (in degrees) curve (using MATLAB and semilogy function with grid) in estimating β parameter as a function of $\beta \in [20^o, 160^o]$.

(d) Repeat the previous part for $10dB$ SNR level.

(e) Suppose instead of one time snapshot, we take two time snapshots (sepatarated by $\frac{1}{F_o}$ so that the s components at two snapshots are identical for each antenna and the phase component). The noises at different time points are assumed to be independent and identical. Obtain the new standard deviation lower bound (for 10dB SNR and 5 antennas) and plot it.

5. *Lower Bound Exercise for a Biased Estimator*

Consider the standard vector estimation setup where $\mathbf{\Theta} \in \Re^m$ is the desired (deterministic) parameters, $\mathbf{x} \in \Re^n$ is the random observation vector whose distribution is given by $f_{\mathbf{x}}(\mathbf{x}; \mathbf{\Theta})$. Suppose $\hat{\mathbf{\Theta}}(\mathbf{x})$ is a biased estimator of Θ, where the bias is represented by

$$b(\mathbf{\Theta}) = E(\hat{\mathbf{\Theta}}(\mathbf{x})) - \mathbf{\Theta}.$$

Find a lower bound for the covariance of this estimator.