Phys 312: Quiz # 3

Fall 2019

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have One hour.
- **Give details of your response to each problem. You will not be given any credit, if it is not clear how you have obtained your response.**
- In the following (x, y, z) are Cartesian coordinates, t is time, $\hat{\mathbf{x}}, \hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ respectively label the unit vectors along the x-, y-, and z-axes, and $\mathbf{r} := x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$.

****

Problem 1 Consider and electromagnetic field configuration in which the magnetic field and scalar potential are give by $\mathbf{B} = \frac{B_0}{\sqrt{1+\alpha^2}}\left[\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}} + \alpha\,\hat{\mathbf{z}}\right]$ and $V = -E_0 z$, where B_0, α, ω, and E_0 are real constants.

1.a (5 points) Find a real number β such that $\mathbf{A} := \beta\mathbf{r}\times\mathbf{B}$ is an admissible vector potential for this field configuration.

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times(\beta\vec{r}\times\vec{B}) = \beta\,\vec{\nabla}\times(\vec{r}\times\vec{B})$$

$$\vec{r}\times\vec{B} = B_1\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ \cos\omega t & \sin\omega t & \alpha \end{vmatrix} \quad \text{where } B_1 := \frac{B_0}{\sqrt{1+\alpha^2}}$$

$$= B_1\left[\hat{x}\underbrace{(\alpha y - z\sin\omega t)}_{=:X} - \hat{y}\underbrace{(\alpha x - z\cos\omega t)}_{=:Y} + \hat{z}\underbrace{(x\sin\omega t - y\cos\omega t)}_{=:Z}\right]$$

$$\vec{\nabla}\times(\vec{r}\times\vec{B}) = \left[\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right)\hat{x} + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right)\hat{y} + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\hat{z}\right]B_1$$

$$= \left[(-\cos\omega t - \cos\omega t)\hat{x} + (-\sin\omega t - \sin\omega t)\hat{y} + (-\alpha - \alpha)\hat{z}\right]B_1$$

$$= -2\frac{\vec{B}}{B_1}B_1 = -2\vec{B}$$

$$\Rightarrow \vec{B} = -2\beta\vec{B} \Rightarrow \beta = -\frac{1}{2}$$

1.b **(10 points)** Find the electric field.

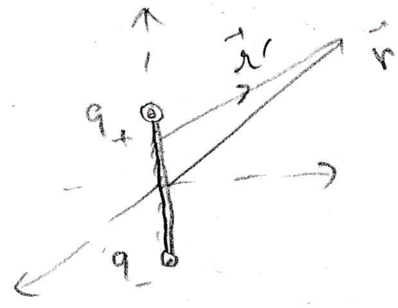
$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}(-E_0 z) - \frac{\partial}{\partial t}\left\{ B_2 \left[\hat{x}(\alpha y - z\sin\omega t) - \hat{y}(\alpha x - z\cos\omega t) + \hat{z}(x\sin\omega t - y\cos\omega t) \right] \right\}$$

where $B_2 := \beta B_1$

$$= E_0 \hat{z} - B_2 \left[\hat{x}(-z\omega\cos\omega t) - \hat{y}(+z\omega\sin\omega t) + \hat{z}(x\omega\cos\omega t + y\omega\sin\omega t) \right]$$

$$= \frac{B_0 \omega}{2\sqrt{1+\alpha^2}} \left[-\hat{x} z\cos\omega t - \hat{y} z\sin\omega t + \hat{z}\left(\frac{2E_0\sqrt{1+\alpha^2}}{B_0 \omega} + x\cos\omega t + y\sin\omega t \right) \right]$$

Problem 2 (15 points) Let q_0, δ, and d be positive real parameters and consider a pair of point charges connected with a wire and having charges $q_\pm := \pm q_0 e^{-t^2/2\delta^2}$ and positions $\mathbf{r}'_\pm := \pm\frac{d}{2}\hat{\mathbf{z}}$. Supposing that the wire corresponds to the line segment connecting the two charges, find the advanced vector potential $\mathbf{A}(\mathbf{r}, t)$ for $t > 0$ and $r \gg d$ in the leading order in d/r.



$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3} \frac{\vec{J}(\vec{r}', t_a)}{\mathscr{r}'}\, d\vec{r}'^3$$

$$= \frac{\mu_0}{4\pi}\int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{\vec{I}(z', t_a)}{\mathscr{r}'}\, dz'$$

$$\vec{I}(z', t) = \frac{dq}{dt}\hat{z} = -\frac{t}{\delta^2} q_0 e^{-t^2/2\delta^2}\hat{z}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-\frac{d}{2}}^{\frac{d}{2}} \left[-\frac{q_0 t_a}{\delta^2} e^{-\frac{t_a^2}{2\delta^2}}\hat{z}\right]\frac{dz'}{\mathscr{r}'}$$

$$t_a = t + \frac{\mathscr{r}'}{c} \qquad \mathscr{r}' = |\vec{r} - z'\hat{z}|$$

$$\Downarrow$$

$$\boxed{\vec{A}(\vec{r},t) = -\left(\frac{\mu_0 q t_0 e^{-\frac{t_0^2}{2\delta^2}}}{4\pi\delta^2}\right)\frac{d}{r}\hat{z} + O\left(\frac{d}{r}\right)^2}$$

where $t_0 := t + \frac{r}{c}$