Math 103 Homework Set # 3[1]

Due on March 28, 2006, at 12:30

1) Let $R : A \to B$ be a relation. The inverse (relation) R^{-1} is defined by

$$R^{-1} := \{(b, a) \in B \times A | (a, b) \in R\}.$$

Prove that

1.1) $Dom(R^{-1}) = Im(R)$;

1.2) $Im(R^{-1}) = Dom(R)$;

1.3) $(R^{-1})^{-1} = R$.

where Dom and Im denote the domain and image of the corresponding relation.

2) Let $R_1 : A \to B$, $R_2 : B \to C$ be two relations. The composite relation $R_2 \circ R_1 : A \to C$ is defined by

$$R_2 \circ R_1 := \{(a, c) \in A \times C | \exists b \in B \ni (a, b) \in R_1 \wedge (b, c) \in R_2\} .$$

Prove that $(R_2 \circ R_1)^{-1} = R_1^{-1} \circ R_2^{-1}$.

3) Let $f : A \to B$ be a function and $R : A \to A$ be the relation defined by

$$R = \{(x, y) \in A^2 | f(x) = f(y)\}.$$

Show that R is an equivalence relation.

4) Let $R : A \to A$ be an equivalence relation. Denote the set of the equivalence classes of R by A/R and let $g : A \to A/R$ be defined by $g(a) = \bar{a}$, where $\bar{a}$ denotes the equivalence class of $a \in A$.

4.1) Prove that g is a function.

4.2) Prove that R can be expressed in the form

$$R = \{(x, y) \in A^2 | f(x) = f(y)\}.$$

5) Let $R : A \to A$ and $S : B \to B$ be partial ordering relations. Let $T : A \times B \to A \times B$ be the relation defined by

$$\forall a_1, a_2 \in A, \quad \forall b_1, b_2 \in B, \quad T = \{((a_1, b_1), (a_2, b_2)) \mid (a_1, a_2) \in R \ \wedge \ [(a_1 = a_2) \Rightarrow (b_1, b_2) \in S]\} .$$

5.1) Prove that T is a partial ordering on $A \times B$.

5.2) Let $A = B = \mathbb{R}$ and R and S be the ordinary ordering of numbers. Is the relation T a total ordering on $\mathbb{R}^2$? Why?

6) Let $\preceq$ be the relation on $\mathbb{R}^2$ defined by

$$\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 , \quad (x_1, y_1) \preceq (x_2, y_2) \ \Leftrightarrow \ x_1^2 + y_1^2 \leq x_2^2 + y_2^2 ,$$

where $\leq$ is the ordinary ordering of real numbers.

6.1) Is $\preceq$ a partial ordering? Why?

6.2) Is $\preceq$ a total ordering? Why?

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[1] Each problem will be graded out of 20 points.