where $P$ is the set of people.

**Problem 5.2** *Let* $R_1$*,* $R_2$*,* $R_3$ *be the relations given in Problem 5.1, determine* $R_1(\{\text{you}\})$*,* $R_2(\{\text{you}\})$*,* $R_3(\{\text{you}\})$*,* $R_1^{-1}(\{\text{you}\})$*,* $R_2^{-1}(\{\text{you}\})$*,* $R_3^{-1}(\{\text{you}\})$*.*

**Problem 5.3** Let $R := \{(m,n) \in \mathbb{Z}^2 \mid 0 < m - n < 3\}$, find $\mathrm{Dom}(R)$, $\mathrm{Ran}(R)$, $R(\{-1,1\})$, and $R^{-1}(\{-1,1\})$.

**Problem 5.4** Let $f := \{(x,y) \in \mathbb{R}^2 \mid y = \frac{1}{\sqrt{x}}\}$ and $I$ be the interval $(-2,4]$. Find $\mathrm{Dom}(f)$, $\mathrm{Ran}(f)$, $f(I)$ and $f^{-1}(I)$.

**Problem 5.5** Let $A$ and $B$ be sets, $C \subseteq A$, $D \subseteq B$, and $R \subseteq A \times B$ be a relation. Show that

(a) If $\mathrm{Dom}(R) \subseteq C$, then $R(C) = \mathrm{Ran}(R)$. In particular, $R(\mathrm{Dom}(R)) = \mathrm{Ran}(R)$.

(b) If $\mathrm{Ran}(R) \subseteq D$, then $R^{-1}(D) = \mathrm{Dom}(R)$. In particular, $R^{-1}(\mathrm{Ran}(R)) = \mathrm{Dom}(R)$.

**Problem 5.6** Let $A$ and $B$ be nonempty sets, and $R \subseteq A \times B$. Show that $\mathrm{Dom}(R) = A$ if and only if $\forall a \in A$, $\exists b \in B$, $a\ R\ b$, i.e., $A \subseteq \mathrm{Dom}(R)$.

**Problem 5.7** Use Definition 5.1.3 to prove that equal relations have equal ranges.

**Problem 5.8** Let $A$ and $B$ be nonempty sets, $R \subseteq A \times B$, $C$ be a nonempty subset of $A$, and $R|_C$ be the restriction of $R$ to $C$. Prove the following statements.

(a) $R|_C \subseteq R$;

(b) $\mathrm{Dom}(R|_C) = C \cap \mathrm{Dom}(R)$;

(c) $\mathrm{Ran}(R|_C) = R(C)$;

(d) If $\mathrm{Dom}(R) \subseteq C$, then $R|_C = R$.

**Problem 5.9** Determine whether the relations given in Problems 5.1, 5.3 and 5.4 are reflexive, symmetric, antisymmetric, or transitive.

**Problem 5.10** Prove that a relation $R$ is symmetric if and only if it is equal to the inverse relation $R^{-1}$.

**Problem 5.11** Let $R$ be a relation. Prove that the following statements are logically equivalent.

$\mathfrak{a} := (R = \varnothing)$;

$\mathfrak{b} := (\mathrm{Dom}(R) = \varnothing)$;

$\mathfrak{c} := (\mathrm{Ran}(R) = \varnothing)$.

**Problem 5.12** Let $A$, $B$, $C$ be nonempty sets, $A' \subseteq A$, and $R \subseteq A \times B$, $S \subseteq B \times C$ be relations. Show that

(a) $(S \circ R = \varnothing) \Leftrightarrow (\mathrm{Dom}(S) \cap \mathrm{Ran}(R) = \varnothing)$;

(b) $(S \circ R)(A') = S(R(A'))$;

(c) $\mathrm{Ran}(S \circ R) = S(\mathrm{Ran}(R))$.

**Problem 5.13** Let $P$ be the set of people, $S := \{(a,b) \in P^2 \mid a \text{ is } b\text{'s sister.}\}$, and $U := \{(a,b) \in P^2 \mid a \text{ is } b\text{'s uncle.}\}$. Determine

(a) the domain and range of $S$ and $U$;

(b) the relations $S^2 := S \circ S$ and $U^2 := U \circ U$;

(c) the relations $S \circ S^{-1}$ and $U \circ U^{-1}$;

(d) the relations $S \circ U$ and $U \circ S$;

(e) the relations $(S \circ U)^{-1}$ and $(U \circ S)^{-1}$;

(f) the relations $S^{-1} \circ U^{-1}$ and $U^{-1} \circ S^{-1}$;

(g) the relation $S \circ U \circ S$ and its domain and range.

**Problem 5.14** Let $R$ be a relation. Show that $\mathrm{Dom}(R^{-1} \circ R) = \mathrm{Ran}(R^{-1} \circ R) = \mathrm{Dom}(R)$.

**Problem 5.15** Let $R$ be a relation. Show that

(a) $\mathrm{Id}_{\mathrm{Dom}(R)} \subseteq R^{-1} \circ R$;

(b) $\mathrm{Id}_{\mathrm{Ran}(R)} \subseteq R \circ R^{-1}$.

**Problem 5.16** Let $R := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x - y \leq 1\}$.

(a) Find the domain of $R$;

(b) Find the image of of the interval $(-1, 1)$ under $R$;

(c) Find the inverse image of the interval $[1, 2]$ under $R$;

(d) Determine if $R$ is reflexive, symmetric, antisymmetric, or transitive.

**Problem 5.17** Let $S$ be a set and $\sim$ be an equivalence relation in $S$. Prove that $S/\!\sim$ is a set.

**Problem 5.18** Let $A$ be a nonempty set, $\{A_\alpha\}_{\alpha \in \mathfrak{B}}$ be a partition of $A$, and

$$\doteqdot \;:=\; \{(a, b) \in A^2 \mid \exists \alpha \in \mathfrak{B}, (a \in A_\alpha) \wedge (b \in A_\alpha)\ \}.$$

(a) Show that $\doteqdot$ is an equivalence relation.

(b) Show that $A/\!\doteqdot\, = \{A_\alpha\}_{\alpha \in \mathfrak{B}}$.

**Problem 5.19** Find all possible equivalence relations in the set $A := \{1, 2, 3, 4\}$.

**Problem 5.20** Let $A_n := \{1, 2, 3, \cdots, n\}$. Determine the number of partitions of $A_2$, $A_3$, $A_4$, $A_5$, $A_6$, and $A_7$. Obtain a formula for the number of partitions of $A_n$ for all $n \in \mathbb{Z}^+$.
*Hint:* Consider the cases of even and odd $n$ separately, and note that for all $m, n \in \mathbb{Z}^+$, with $m \leq n$, there are $\binom{n}{m} := \frac{n!}{m!(n-m)!}$ ways to choose $m$ objects out of $n$ objects.

**Problem 5.21** Show that the causal relation (5.7) is a partial ordering relation which is not a total ordering.

**Problem 5.22** Let $S$ be a nonempty set and $R \subseteq S^2$ be a reflexive and transitive relation.

(a) Show that $\sim := \{(a, b) \in S^2 \mid (a\ R\ b) \wedge (b\ R\ a)\ \}$ is an equivalence relation.

(b) Show that $\preccurlyeq := \{(A, B) \in (S/\!\sim)^2 \mid \exists a \in A, \exists b \in B,\ a\ R\ b\ \}$ is a partial ordering relation.

(c) Show that if for all $a, b \in S$ either $a\ R\ b$ or $b\ R\ a$, then $\preccurlyeq$ is a total ordering.