Solutions.

# Math 103: Quiz # 3

**Fall 2007**

- Write your name and Student ID number in the space provided below and sign.

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|---|---|
| **Name, Last Name:** | |
| **ID Number:** | |
| **Signature:** | |

- You have 50 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

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**1.** Give the definition of the following terms. See the book.

**1.a)** Inductive set: (5 points)

**1.b)** $\mathbb{N}$: (5 points)

**1.c)** Domain of a relation. (5 points)

**1.d)** Image of a set $X$ under a relation $R$. (5 points)

**2.** Give the definition of $+$ and $\leq$ for natural numbers and use them to prove that $\forall m, n \in \mathbb{N}$, $m \leq m+n$. (30 points)

$\forall m, n \in \mathbb{N}$, $\quad m + 0 := m \quad$ and $\quad m + S(n) := S(m+n)$.

$\forall m, n \in \mathbb{N}$, $\quad (m \leq n) := (m \subseteq n)$.

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$\forall n \in \mathbb{N}$. let $a_n := (\forall m \in \mathbb{N},\ m \leq m+n)$.

- For $n = 0$, $\quad m + n = m + 0 = m \quad$ and $\quad m \subseteq m = m+n \Rightarrow m \leq m+n$.

So $a_0$ holds.

- Suppose $\exists k \in \mathbb{N}$, $a_k$, i.e., $\forall m \in \mathbb{N}$, $m \leq m+k$
- Prove $a_{k+1}$: $\forall m \in \mathbb{N}$, $m + (k+1) = m + S(k) = S(m+k)$

We know that for any set $A$, $S(A) = A \cup \{A\}$

So $A \subseteq S(A) \Rightarrow m + k \subseteq S(m+k) = m + (k+1) \quad (1)$

By induction hypothesis: $m \leq m+k \Rightarrow m \subseteq m+k \quad (2)$

(1) & (2) $\Rightarrow (m \subseteq m+k) \wedge (m+k \subseteq m + (k+1))$

$\Rightarrow m \subseteq m + (k+1) \Rightarrow m \leq m + (k+1)$. This proves $a_{m+1}$.

By induction this implies $\forall n \in \mathbb{N}, a_n$. ∎

**3.** Let $A$ and $B$ be nonempty sets, $R \subseteq A \times B$, and $\{A_\alpha\}_{\beta \in \mathfrak{B}}$ be a family of subsets of $A$. Prove that $R(\bigcup_{\beta \in \mathfrak{B}} A_\beta) \subseteq \bigcup_{\beta \in \mathfrak{B}} R(A_\beta)$. (20 points)

$\forall b \in R(\bigcup_{\beta \in \mathfrak{B}} A_\beta)$, $\exists a \in \bigcup_{\beta \in \mathfrak{B}} A_\beta$, $a R b$

$a \in \bigcup_{\beta \in \mathfrak{B}} A_\beta \Rightarrow \exists \beta_* \in \mathfrak{B}$, $a \in A_{\beta_*}$ $\Rightarrow b \in R(A_{\beta_*})$

$\Rightarrow b \in \bigcup_{\beta \in \mathfrak{B}} R(A_\beta)$

But $R(A_{\beta_*}) \subseteq \bigcup_{\beta \in \mathfrak{B}} R(A_\beta)$

$\Rightarrow R(\bigcup_{\beta \in \mathfrak{B}} A_\beta) \subseteq \bigcup_{\beta \in \mathfrak{B}} R(A_\beta)$ ∎

**4.** Let $A := \{2\}$ and $R := \{(m,n) \in \mathbb{Z}^{+2} \mid m|(2n^2-1)\}$.

**4.a)** Find the image of $A$ under $R$. (10 points)

$$R(A) = \{n \in \mathbb{Z}^+ \mid \exists m \in A,\ m \mid (2n^2-1)\}$$
$$= \{n \in \mathbb{Z}^+ \mid 2 \mid 2n^2-1\}$$

If $n \in R(A)$, $2 \mid 2n^2-1$ but this is impossible because $2n^2-1$ is odd. This shows that $R(A)$ has no elements, i.e., $R(A) = \phi$.

**4.b)** Find the inverse image of $A$ under $R$. (10 points)

$$R^{-1}(A) = \{m \in \mathbb{Z}^+ \mid \exists n \in A,\ m \mid 2n^2-1\}$$
$$= \{m \in \mathbb{Z}^+ \mid m \mid 7\}$$
$$= \{1, 7\}$$

$(2(2^2)-1=7$

**4.c)** Is $R$ symmetric? Why? (10 points) No, because.

$1 \mid 7 = 2(2^2)-1 \Rightarrow 1R2 \Rightarrow (1,2) \in R$.

But $2(1^2)-1 = 1$ and $2 \nmid 1 \Rightarrow (2,1) \notin R$.