Math 103: Quiz # 7

Fall 2007

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 50 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

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1. Give the definition of the following terms.

1.a) invertible function: (5 points) See The Book.

1.b) a transposition of I_n: (5 points)

1.c) a permutation of I_n: (5 points)

2. Let τ_1 and τ_2 be the transpositions of I_3 that are given by $\tau_1 := (1 \leftrightarrow 2)$ and $\tau_2 := (2 \leftrightarrow 3)$. Determine all the permutations of I_3 and express them in terms of τ_1 and τ_2. (30 points)

- $Id_{I_3} = \tau_1 \circ \tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$
- $\tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$
- $\tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$
- $\tau_1 \circ \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$
- $\tau_2 \circ \tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$
- $\tau_1 \circ \tau_2 \circ \tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \tau_2 \circ \tau_1 \circ \tau_2$

Observe that $\tau_1 \circ \tau_2 \circ \tau_1$ is the remaining transposition of I_3.

By Thm 6.5.1, σ is a permutation of I_3 iff it's a bijection. It's easy to see that there are only 6 bijections from I_3 to itself. Therefore, there does not exist any other permutation of I_3.

3. Let A, B be nonempty sets and $f : A \to B$ be an everywhere-defined and one-to-one function. Prove that $\phi : A \to \text{Ran}(f)$ defined by $\forall a \in A$, $\phi(a) := f(a)$ is a bijection. (25 points)

Let's see that ϕ is everywhere defined. Let $a \in A$. Then, $\exists\, b \in \text{Ran} f$ s.t. $f(a) = b$ since f is everywhere defined. So, $\phi(a) = f(a) \in \text{Ran} f$, i.e. ϕ is defined at a. But a was arbitrary, so ϕ is everywhere defined.

Let $a, a' \in A$. We know that f is 1-1 and well-defined, therefore

$a = a' \Leftrightarrow f(a) = f(a')$. On the other hand,

$f(a) = f(a') \Leftrightarrow \phi(a) = \phi(a')$, because $f(a) = f(a') \in \text{Ran} f$.

Consequently, $a = a' \Leftrightarrow \phi(a) = \phi(a')$, i.e. ϕ is well-defined and 1-1.

Let $b \in \text{Ran} f$. Then, $\exists\, a \in A$ s.t. $f(a) = b$. Or, $\phi(a) = f(a)$, so $\phi(a) = b$.

So, $b \in \text{Ran}\, \phi$. This means ϕ is onto.

Finally, ϕ is a bijection.

4. Let $n \in \mathbb{Z}^+$. Prove that every transposition of I_n is a bijection (30 points)

See Prop 6.5.1. on page 111.