Problems

Problem 2.1 Specify the truth value of the following statements and determine their negation.

$$\mathfrak{a}_1 := (\forall x \in \mathbb{N},\ \exists y \in \mathbb{N},\ x < y)$$

$$\mathfrak{a}_2 := (\exists y \in \mathbb{N},\ \forall x \in \mathbb{N},\ x < y)$$

$$\mathfrak{a}_3 := (\exists x \in \mathbb{N},\ \forall y \in \mathbb{N},\ x < y)$$

$$\mathfrak{a}_4 := (\forall y \in \mathbb{N},\ \exists x \in \mathbb{N},\ x < y)$$

$$\mathfrak{a}_5 := (\exists x \in \mathbb{N},\ \exists y \in \mathbb{N},\ x < y)$$

$$\mathfrak{a}_6 := (\forall x \in \mathbb{N},\ \forall y \in \mathbb{N},\ x < y)$$

Problem 2.2 Let $\mathfrak{p}(x)$ be a predicate depending on a variable x that takes values in a set A. Express the negation of the statement "$\exists! x \in A, \mathfrak{p}(x)$" using mathematical symbols.

Problem 2.3 Obtain a solution of Exercise 2.5.3 and a proof of part (b) of Theorem 2.6.1 by constructing the relevant truth tables.

Problem 2.4 Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be statements. Prove that $(\mathfrak{a} \Leftrightarrow \mathfrak{b}) \Leftrightarrow \mathfrak{c}$ is logically equivalent to $\mathfrak{a} \Leftrightarrow (\mathfrak{b} \Leftrightarrow \mathfrak{c})$, i.e., $\Leftrightarrow$ is an associative operation.

Problem 2.5 Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be statements. Determine if the following compound statements are tautologies.

$$\mathfrak{d}_1 := (\mathfrak{a} \Leftrightarrow (\mathfrak{b} \wedge \mathfrak{c})) \Leftrightarrow ((\mathfrak{a} \Leftrightarrow \mathfrak{b}) \wedge (\mathfrak{a} \Leftrightarrow \mathfrak{c}))$$

$$\mathfrak{d}_2 := (\mathfrak{a} \Leftrightarrow (\mathfrak{b} \vee \mathfrak{c})) \Leftrightarrow ((\mathfrak{a} \Leftrightarrow \mathfrak{b}) \vee (\mathfrak{a} \Leftrightarrow \mathfrak{c}))$$

$$\mathfrak{d}_3 := (\mathfrak{a} \Leftrightarrow (\mathfrak{b} \Rightarrow \mathfrak{c})) \Leftrightarrow ((\mathfrak{a} \Leftrightarrow \mathfrak{b}) \Rightarrow (\mathfrak{a} \Leftrightarrow \mathfrak{c}))$$

Problem 2.6 Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}', \mathfrak{b}'$ be statements such that $\mathfrak{a} \Leftrightarrow \mathfrak{a}'$ and $\mathfrak{b} \Leftrightarrow \mathfrak{b}'$. Prove that the following compound statements are tautologies.

$$\mathfrak{c}_1 := ((\mathfrak{a} \wedge \mathfrak{b}) \Leftrightarrow (\mathfrak{a}' \wedge \mathfrak{b}')).$$

$$\mathfrak{c}_2 := ((\mathfrak{a} \vee \mathfrak{b}) \Leftrightarrow (\mathfrak{a}' \vee \mathfrak{b}')).$$

$$\mathfrak{c}_3 := ((\mathfrak{a} \Rightarrow \mathfrak{b}) \Leftrightarrow (\mathfrak{a}' \Rightarrow \mathfrak{b}')).$$

Problem 2.7 Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ be statements, $\mathfrak{d} := (\neg\mathfrak{a} \Rightarrow (\mathfrak{b} \Rightarrow \mathfrak{c}))$, and $\mathfrak{e} := (\neg(\mathfrak{a} \Rightarrow \mathfrak{b}) \Rightarrow \mathfrak{c})$. Determine whether $\mathfrak{d} \Rightarrow \mathfrak{e}$ is a tautology?

Problem 2.8 Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ be statements, $\mathfrak{d} := ((\mathfrak{b} \Rightarrow \mathfrak{a}) \Rightarrow (\mathfrak{b} \wedge \mathfrak{c}))$, and $\mathfrak{e} := (\mathfrak{b} \wedge (\mathfrak{a} \Rightarrow \mathfrak{c}))$. Show the logical equivalence of $\mathfrak{d}$ and $\mathfrak{e}$ by

(a) constructing the corresponding truth table;

(b) using the methods of propositional calculus.

Problem 2.9 Repeat Problem 2.8 for $\mathfrak{d} := (\mathfrak{a} \wedge (\mathfrak{b} \Rightarrow \neg\mathfrak{a}))$ and $\mathfrak{e} := (\neg(\mathfrak{a} \Rightarrow \mathfrak{b}))$.

Problem 2.10 Let $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be statements. For each of the following statements find a logically equivalent statement that only involves $\neg$ and $\vee$.