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Preface

Learning abstract mathematics is a slow and complex process. This is partly because a student of mathematics must not only try to grasp the ideas and master the techniques of approaching mathematical problems, but she must also learn the language of mathematics. This is often a cause of unease for many students in their first serious course in mathematics. The present text aims to help such students. The only prerequisite for its effective use is a basic knowledge of arithmetic and high school algebra.

Another aim of this book is to provide the reader with a global view of mathematics. For example, it includes a description of the methodology of mathematics and the rules governing its development. In particular, it discusses the issue of how one generalizes the known mathematical concepts.

This book has developed out of a set of lecture notes that I prepared while teaching a freshman level course, entitled: "Introduction to Abstract Mathematics," at Koç University between 1998 and 2006. In this course I usually followed C. Schumacher's "Chapter Zero" (both editions 1 and 2.) Therefore, most of the topics I have covered in the present text are identical with those of "Chapter Zero." The approach I have pursued in presenting these topics, however, is quite different. In particular, I have tried to avoid leaving gaps in the proofs of the theorems. In some cases, I give a sketch of the proof to outline the underlying strategy and then present the complete proof to demonstrate how a proof is properly composed. I have also made a special effort to make the proofs as readable as possible.

Chapter 1 is a general introduction to the structure of mathematics and its similarities and differences with natural sciences. Chapter 2 discusses the elements of logic and propositional calculus. Chapter 3 describes various theorem types and proof methods. Chapter 4 introduces the elementary axioms of set theory and elucidates their importance in developing a consistent treatment of sets. Chapters 5 and 6 offer a thorough discussion of relations and functions, respectively. Here I have paid special attention to eliminate any ambiguity related with the domain of relations and functions. Chapter 7 motivates and explores the notion of the cardinality of a set. Finally, Chapter 8 provides a survey of mathematical theories and their common features.

Except Chapter 1, each chapter ends with a list of mostly elementary exercise problems. Some of these have direct applications in the development of the subject. Therefore, I urge the reader to work out as many of these problems as possible.

The book includes five appendices. These are devoted to topics that are either more technical in nature or whose inclusion in the text could divert the attention of the reader from the conceptually more important issues.

Koç University, Istanbul, September 2007 — Ali Mostafazadeh

Chapter 1

Introduction

1.1 General Structure of Mathematics

Mathematics may be crudely described as the study of a collection of abstract concepts that have been introduced primarily for practical reasons. These concepts and their interrelationships form a body of knowledge that is gathered throughout the history of mankind and is continuously perfected and passed on from generation to generation.

Mathematics is usually divided into two disciplines: *Abstract* or *Pure Mathematics* and *Applied Mathematics*. What makes mathematics so important in our everyday lives is its applications in various areas of natural sciences and engineering. "Applied Mathematics" is a branch of Mathematics that deals with the mathematical methods with immediate practical applications. There are many examples of mathematical theories that were initially considered to be void of useful applications but that later found to serve important practical purposes. In this sense "Abstract Mathematics" might be identified with the branch of mathematics whose practical significance is not presently well-understood or appreciated.

An important advantage of mathematics over other scientific disciplines is that it has an extremely precise and concise language. Every mathematical study deals with a collection of basic objects (words of the language) that are described in terms of their primary and secondary properties. The primary properties are given in the form of ***definitions*** as well as a short list of statements called ***axioms*** that are accepted as being true. The secondary properties are then derived through logical reasoning from the definitions and axioms. It is these secondary properties that are usually considered as ***mathematical knowledge***.

In principle, there is no reason why a mathematician should define a new mathematical object or adopt an axiom the way she does. The only limitations are that *a definition must be precise and an axiom must be consistent with the other axioms and independent from them*, i.e., it should neither contradict nor follow from the other axioms. In short, a statement about the objects under study is a candidate for a new axiom if and only if one cannot prove or disprove its validity based on the previously adopted axioms. Once such a statement is found the mathematician is absolutely free to choose one of the following three options:

1. To promote the statement to the status of a new axiom, i.e., accept it as being true;

2. To take the opposite statement as a new axiom, i.e., accept that the statement is false;

3. To take no action, i.e., leave the question of the validity of the statement unanswered.

One of the most important discoveries of the twentieth century is the celebrated "Incompleteness Theorem" of Kurt Gödel (1906-1978). This theorem asserts that for any given finite list of axioms, regardless of their content, there exists a statement whose validity cannot be decided using the adopted axioms, i.e., there are always candidates for new axioms. Such candidates are however very difficult to be discovered. Indeed much of the mathematical activity concerns finding useful statements, called ***theorems***, that follow as logical consequences of the adopted axioms rather than searching for undecidable statements. Theorems that are primarily used in proving other theorems are called ***lemmas***. Theorems that follow as immediate consequences of a preceding theorem are called ***corollaries***. Theorems of limited and central importance for the development of the subject are respectively called ***propositions*** and ***fundamental theorems***.

In addition to the adopted definitions and axioms and the proven theorems there is a third category of mathematical statements called ***conjectures***. These are statements which are expected to acquire the status of a theorem in future, i.e., one is unable at present to prove or disprove them but at the same time cannot prove that they are independent of the axioms.

A ***mathematical theory*** is the collection of the definitions, axioms, theorems (including lemmas, propositions, corollaries), and conjectures. The oldest of all mathematical theories is *Number Theory*. The objects of study are positive integers: $1, 2, 3, \cdots$ and their arithmetic properties. One of the most famous theorems of Number Theory and indeed mathematics generally is the so-called *Fermat's last theorem*. It asserts that *there are no positive integers* a, b, *and* c *satisfying* $a^n + b^n = c^n$, *if* n *is an integer greater than 2*. This theorem was discovered by the French mathematician Pierre de Fermat (1601-1665) in the seventeenth century but resisted a proof for about 350 years. It was considered as a conjecture of primary importance until 1994 when it was finally proven by Andrew Whiles of the Princeton University. Currently, there are dozens of conjectures awaiting proofs. Some of these like the *Riemann Hypothesis* are the subject of million-dollar prizes.

All mathematical theories have the above-mentioned general structure. At the very foundation there lie the basic definitions and axioms. These lead to a growing number of theorems that form our knowledge of the theory and facilitate its application in other areas. This simple structure of mathematical theories have a rather mysterious aspect: *There is no clue as to how to select the axioms of a theory.* One can raise the same question for the definitions. In reality the definition of the objects and some of the more basic axioms follow from human intuition and usually one does not even entertain the idea of modifying them. But there are instances where an axiom seems not so intuitive. In principle, one can suppose, as an alternative axiom, that it is false. This leads to a new theory which is not compatible with the former theory but equally acceptable. The fact that we do not encounter many instances of rival theories of this sort is indeed surprising. The best example of such rival theories are *Euclidean* and *non-Euclidean geometries*.

The Euclidean plane geometry fully worked out in ancient Greece and precisely formulated by Euclid in his book *Elements* about 300 BC is the mathematical theory concerning geometric objects in a plane, e.g., points, lines, circles, triangles, etc. The axioms of this theory are the following:

$\mathfrak{e}_1$) Two points determine a line.

$\mathfrak{e}_2$) A line segment can be extended indefinitely.

$\mathfrak{e}_3$) A point and a radius (line segment) determine a circle.

$\mathfrak{e}_4$) Any two right angles are equal.

$\mathfrak{e}_5$) Given a line L and a point p not lying on L, there exists one and only one line passing through p that is parallel to L.

The first four of Euclid's axioms are quite basic and intuitively acceptable. The fifth axiom is rather different. This apparent difference in character led several generations of mathematicians to seek whether the fifth axiom could be eliminated from the above list by proving that it follows from the other four. This question took no less than twenty two centuries to be answered satisfactorily. It turned out that the fifth axiom was actually independent. More importantly, in trying to establish this fact it was realized that the fifth axiom was not indispensable for formulating geometric theories. Replacing $\mathfrak{e}_5$ with either of its negations, namely

$\mathfrak{e}_5'$) Given a line L and a point p not lying on L, there exist more than one line passing through p that is parallel to L;

$\mathfrak{e}_5''$) Given a line L and a point p not lying on L, there exists no line passing through p that is parallel to L,

leads to two different non-Euclidean geometries. These are respectively the *hyperbolic geometry* of Nikolai Lobachevsky (1792-1856) and Jánus Bolyai (1802-1860) and the *spherical geometry* of Bernhard Riemann (1826-1866).[1]

The discovery of non-Euclidean geometry was initiated primarily to settle the abstract mathematical question of the independency of Euclid's fifth axiom. This discovery was indeed a natural outcome of the attempt initially made by Giovanni Saccheri (1667-1733) much before Lobachevsky, Bolyai, and Riemann to prove the independency and hence the necessity of the fifth axiom by considering the consequences of its negation. About 75 years after the discovery of non-Euclidean geometry, Albert Einstein (1879-1955) identified non-Euclidean geometry as the appropriate mathematical tool for his *General Theory of Relativity* also known as *Einstein's Theory of Gravity*. The surprising and highly counterintuitive predictions of this theory have since passed numerous experimental tests. General Theory of Relativity describes the physical laws governing our universe and its content to a great degree of precision.

1.2 Relationship with Natural Sciences

The aim of natural sciences is to understand and control natural phenomena whenever possible. Scientific theories are analogous to mathematical theories. They consist of a collection of basic objects of study with certain primary properties. These consists of some basic defining properties such as the position and mass of a particle in a physical theory and a set of laws, usually called ***postulates*** of the theory, that describe the general relations between these basic properties. The postulates of scientific theories play the same role as the axioms of mathematical theories.

Among the simplest and most powerful scientific theories is *Newtonian mechanics*. The object of the study are particles (point masses) and the postulates are the three laws of Isaac Newton (1642-1727). Another example is *Newton's theory of gravity* that is based on his three laws of mechanics and his law of gravitation.

The analogs of the mathematical theorems are the predictions of a scientific theory. Each set of postulates lead to a set of predictions that follow from the postulates by logical reasoning. A good example is the prediction of the orbits of planets using Newton's theory of gravity.

The main difference between natural sciences and mathematics is that in mathematics the validity of the theorems cannot be disputed after a valid proof is given. A set of axioms leads to a set of theorems and an improvement of a theory is possible only if one discovers new conjectures, proves the known conjectures (i.e., invents new theorems), or expands the axioms by promoting one or more independent statements to new axioms. In natural sciences once a prediction is made it is put to the test of experiment. If an experiment refutes a prediction of a scientific theory, the theory is invalidated. This means that at least one of the postulates of the theory is not correct. In such a case, one attempts at formulating a new theory by changing some or all of the postulates of the old theory. For example, in 1915-1919 Newton's theory of gravity was replaced by Einstein's general theory of relativity, and in 1925-1926 Newtonian mechanics was replaced by *quantum mechanics*.

In reality a scientific theory can never be proven correct, because experiments can only falsify a theory. *The false theories are replaced by those waiting to be falsified !* Therefore, a scientific theory can only be called "*successful*," and the measure of its success is the precision of the most precise experiment that fails to falsify the theory. Usually successful theories lead to

[1] It is believed that Gauss had independently discovered non-Euclidean geometry even before Lobachevsky and Bolyai. He had, however, not sought for the publication of his findings on this subject and only entered them in a series of unpublished notes that were discovered after his death.

predictions that bring about technological advances. These in turn lead to the design of more precise experiments and provide means for a better assessment of the theory. The history of natural sciences shows that as technological advances lead to more precise measuring devices the old theories are gradually falsified and replaced by more successful theories which in turn result in more advanced technologies and more precise experiments. The cycle depicted in Figure 1.1 is the key for the development of the natural sciences and consequently the human civilization.

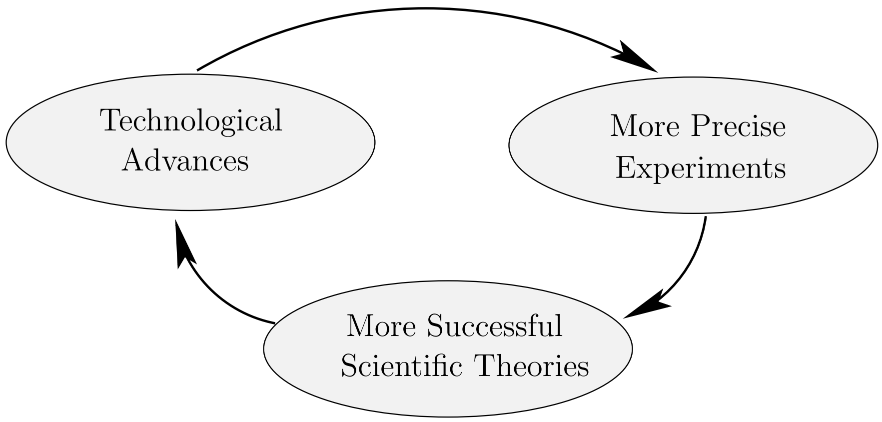

Figure 1.1: Development of Natural Sciences

The situation is different in dealing with mathematical theories, for they cannot be falsified. They can be rejected in favor of others only for practical reasons. In mathematics, there is no external verdict to forbid an axiom and the mathematician is free to make her choice.[2] In contrast, in natural sciences, it is the nature that chooses its laws. The only freedom is left for the scientist is to choose the most appropriate means to discover and exploit these laws. In a sense pursuing mathematics is a much more difficult endeavor than pursing natural sciences, because mathematicians do not have the valuable input of experimental data to shape and perfect their axiomatic systems. Yet, it is this very mathematics that is used so successfully in natural sciences. This is indeed one of the most remarkable mysteries of our universe. It is best described by Eugene Winger (1902-1995) as "*the unreasonable effectiveness of mathematics in natural sciences*."

1.3 Mathematical Language

Perhaps the most important asset of mathematics that distinguishes it from the other areas of knowledge and makes it so powerful is the language it employs. This is a langauge in which every word is assigned a single and precise meaning. A properly composed mathematical text consists of a series of assertions each of which are explicitly identified as definitions, axioms, theorems, and conjectures. The proofs of theorems use the axioms and previously proven theorems in the text or elsewhere and are based on logical reasoning. Consequently, the results reported in a mathematical text cannot be subject to any sort of interpretation. This does not mean that they must be considered as valid, for they may be based on incomplete or logically unacceptable arguments. The main advantage of having a concise and precise language is in detecting such "wrong" arguments.

Like any other language, mathematical language has gradually evolved during the history and it was not up until the mid nineteenth century that it took its present form particularly in the hands of Carl Friedrich Gauss (1777-1855), the *Prince of Mathematics*. Gauss held the view that mathematical langauge must be absolutely precise and that mathematical results

[2]Georg Cantor (1845-1918), the founder of Set Theory, once said: "The essence of mathematics lies precisely in its freedom."

Symbol	Meaning
$=$	The left-hand side is equal to the right-hand side.
$\neq$	The left-hand side is not equal to the right-hand side.
$:=$	The left-hand side is defined to be the right-hand side.
$=:$	The right-hand side is defined to be the left-hand side.
$\in$	The left-hand side is an element of the right-hand side.
$\notin$	The left-hand side is not an element of the right-hand side.
$\subseteq$	The left-hand side is a subset of the right-hand side.
$\mathbb{N}$	Set of natural numbers $:= \{0, 1, 2, 3, \cdots\}$
$\mathbb{Z}$	Set of integers $:= \{\cdots, -2, -1, 0, 1, 2, \cdots\}$
$\mathbb{Z}^+$	Set of positive integers $:= \{1, 2, 3, 4, \cdots\}$
$\mathbb{Z}^-$	Set of negative integers $:= \{-1, -2, -3, -4, \cdots\}$
$\mathbb{Q}$	Set of rational numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set of positive real numbers

Table 1.1: Some basic mathematical symbols and their meaning

must be based on rigorous foundations. The price one pays for promoting mathematical rigor to the extent that Gauss and the following generations of mathematicians did is the difficulty in the use of the resulting mathematical language. However, once one overcomes this difficulty, which is only possible by acquiring a solid mathematical education, the result is most rewarding. One can draw a clear line between the presently available knowledge and the conjectures and speculations surrounding it.

Use of a precise language and rigorous methods is not so much essential in natural sciences, because the ultimate judgement of every scientific theory is its experimental test(s). In mathematics there is no external measure of the validity of a theory. Therefore one must make every effort to ensure its internal consistency.

The process of learning ***mathematical language*** is similar to that of other languages. It ***cannot be mastered unless the student begins practicing it.*** Reading mathematical texts and attending mathematical lectures are integral parts of the learning process. But they are generally futile, if the student does not practice mathematics actively.

Mathematical language makes use of a number of symbols that denote certain commonly used concepts. It also uses words from other languages, in our case English. The mathematical symbols and the technical words borrowed from other languages must be defined precisely. In particular, the meaning of the technical words must not be confused with those provided in typical dictionaries. As we describe below there are a few well-known exceptions to this rule.

We have listed some basic mathematical symbols in Table 1.1 that we will use throughout this book. The first two of these are the equality and inequality symbols, "$=$" and "$\neq$," respectively. We must warn the reader that ***the notion of equality must be defined within each mathematical theory*** and that ***an equation cannot be qualified as true or false unless we specify the identity of both its left- and right-hand sides.*** We will use the defining symbols: "$:=$" and "$=:$" for this purpose and will ***always distinguish between equations***

($a = b$) and defining relations ($a := b$ and $a =: b$).

In our investigation we will encounter mathematical objects that we can consider as a whole. We will use the term "***set***" to describe the totality of these objects. One of the main purposes of this book is to make the notion of a set more precise. We will initially identify a set with a collection of objects that we will call ***elements*** of the set. This is not a genuine definition of a set, for we have not defined in precise terms what we mean by a "*collection*" or an "*object*." We will rely on the reader's basic understanding of these terms and consider them as "*elementary*" in the sense that within the context of our study they do not require a precise description.

Indeed, we are forced into building our understanding of mathematics upon these elementary notions, because in order to define or even describe a mathematical object we need to use other words that stand for more elementary notions. We can continue asking about the definition of these more elementary notions and try to find even more elementary notions to describe them. As we see no point in producing an endless list of more and more elementary notions, at some stage we must be content with our understanding of certain, preferably few, "most elementary" notions and begin using them to describe more sophisticated and consequently more interesting and useful notions.

The reliance on elementary notions is a common feature of all scientific disciplines. For example, in chemistry the notion of an atom is elementary, for a chemist does not enquire into the internal structure of atoms. In mathematics, we start with elementary notions of "collection" and "object" and gradually construct set theory. Sets provide the strong foundations upon which we can build more sophisticated and useful mathematical objects and theories. For examples we can build numbers as certain sets and identify the notion of adding two numbers with yet another set.

A set is a collection of objects, but as we will discover later not every collection of objects is a set! A collection of objects is a set if it possesses certain basic properties. These are the axioms of Set Theory. We will describe most of them as we learn more and more about sets in this book. We will encounter examples of collections of objects that fail to satisfy some of these axioms and hence cannot be called sets. However, for pedagogical reasons we will use the term "set" before discussing the axioms of set theory. This is actually how Set Theory has developed historically.

We will use the ***inclusion symbol*** "$\in$" to state that an object belongs to a collection or that it is an element of a set. For example, "$1 \in \mathbb{R}$" means "1 is an element of the set of real numbers." The symbols for various sets of numbers are listed in Table 1.1. We have also listed in this table the symbol "$\subseteq$" that stands for "is a subset of." *Given two sets A and B, we say that A is a* **subset** *of B and write $A \subseteq B$, if every element of A is an element of B.* That is $a \in A$ implies $a \in B$. The latter statement is a logical implication. Every mathematical analysis is based on logical reasoning. This makes *Logic* an indispensable ingredient of mathematics. We will therefore begin our study by exploring elementary logic in Chapter 2 where we will introduce some other mathematical symbols.

Chapter 2

Elements of Logic

2.1 Statements and Predicates

The building blocks of logical arguments are certain assertions called statements. *A **statement** is an assertion (sentence) that is either true or false.*[1] This means that *in order to establish that a statement is true it is sufficient to show that it is not false.* If a statement is true we say that its ***truth value*** is "True" (T). Similarly, a false statement has truth value "False" (F). The assertion that "1 is less than 2." is a an example of a statement with truth value "T." The assertion that "Every appel is red." is an example of a statement with truth value "F."

There are also assertions that are not statements, e.g., "x is an integer" which we may also express as "$x \in \mathbb{Z}$." This is an example of a predicate. *A **predicate** is an assertion involving one or more variables such that choosing a value for each of the variables turns the assertion into a statement.* For example, setting $x = 2$ in the above predicate turns it into a true statement ($2 \in \mathbb{Z}$), whereas setting $x = \frac{1}{2}$ turns it into a false statement ($\frac{1}{2} \in \mathbb{Z}$). An example of a predicate involving three variables is "$x \in \mathbb{N}, y \in \mathbb{Z}, \epsilon \in \mathbb{R}^+, x - y < \epsilon$." [2] We cannot decide if it is true unless we are provided with further information about the variables x, y and ϵ.

In studying logic we often deal with statements whose content is not specified. These are not to be confused with predicates. An unspecified statement $\mathfrak{a}$ is distinguished from other types of assertions by the conditions that it does not involve any variables and that it is either true or false; there is no other option for the truth value of a statement. The same holds for a predicate, but we are not able to determine the truth value of a predicate. We will also encounter predicates whose variable(s) are unspecified statements.

Definition 2.1.1 *Two statements $\mathfrak{a}$ and $\mathfrak{b}$ are said to be **equal** if they have the same meaning, i.e., they can be used interchangeably in every argument. In this case we write $\mathfrak{a} = \mathfrak{b}$.* ■ [3]

This notion of equality of statements is not quite essential, because "$\mathfrak{a} = \mathfrak{b}$" simply means that $\mathfrak{a}$ is another symbol for $\mathfrak{b}$. Therefore we can completely avoid using "=" if we employ a unique notation for each statement appearing in a logical argument. In contrast, we always need to use the defining symbol ":=" (or "=:") whenever we introduce a new statement.

Consider the following logical argument concerning a statement $\mathfrak{a}$.

$\mathfrak{b}$:="If the statement that '$\mathfrak{a}$ is false' is false, then $\mathfrak{a}$ is true."

The reader undoubtedly agrees with the validity of this argument and that it is true independently of whether $\mathfrak{a}$ is itself true or false. For example, we may identify $\mathfrak{a}$ with the statement: "$1 > 2$." Then

$\mathfrak{b}$="If the statement that '$1 > 2$ is false' is false, then "$1 > 2$" is true."

[1] Some books use the term "proposition" for what we call a "statement."
[2] Here we use the usual symbol "$<$" for "is less than."
[3] We use ■ to mark the end of a definition, proof, or a solution.

Clearly, although $\mathfrak{a}$ is false, $\mathfrak{b}$ is true. As this example shows a valid logical argument, namely $\mathfrak{b}$ is always independent of the details of the circumstances it is applied to. Every logical argument is indeed a statement and what is important is its truth value. Indeed, we may describe Logic as a collection of rules that are used to deal with various statements without having a bearing on the details of their content but only their truth value. This is the main justification for the following definition.

Definition 2.1.2 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be statements. Then $\mathfrak{a}$ is said to be* **logically equivalent** *to $\mathfrak{b}$ if $\mathfrak{a}$ and $\mathfrak{b}$ have the same truth value. In this case we write $\mathfrak{a} \Leftrightarrow \mathfrak{b}$.* ■

A trivial example of two logically equivalent statements is any statement $\mathfrak{a}$ and the statement "$\mathfrak{a}$ is true." We will encounter many nontrivial and useful examples of logically equivalent statements in the following sections.

The notions of equality and logical equivalence for statements may be extended to predicates.

Definition 2.1.3 *Two predicates are said to be* ***equal*** *if they have the same meaning, in particular they depend on the same variable(s). Two predicates are said to be* ***logically equivalent*** *if they depend on the same variable(s) and for each value of the variable(s) they yield logically equivalent statements.* ■

2.2 Qualifiers

As we explained in the preceding section, fixing a particular value for the variable(s) of a predicate yields a statement. This is not the only way of turning a predicate into a statement. We can supplement the predicate with certain qualification of its variables. For example consider the predicate "x is greater than 1." We can turn this into a statement by qualifying x to be an arbitrary integer: "For all integer x, x is greater than 1." This is a false statement. Next consider qualifying x in a different way: "There is an integer x such that x is greater than 1." Clearly, this is a true statement. We will use the symbols "$\forall$" to mean "***for all***" and "$\exists$" for "***there exists one or many***." These are our basic ***qualifiers***. Using these symbols and the usual symbol "$>$" for "is greater than," we can express the preceding two statements as "$\forall x \in \mathbb{Z}$, $x > 1$" and "$\exists x \in \mathbb{Z}, x > 1$," respectively.

A simple property of ***qualified variables*** is that they can be freely relabelled; they ***are dummy variables***. For example, "$\forall x \in \mathbb{Z},\ x > 1$" and "$\forall y \in \mathbb{Z},\ y > 1$" are equal statements. Similarly, we have: $(\forall x \in \mathbb{Z},\ x > 1) = (\forall y \in \mathbb{Z},\ y > 1)$.

In qualifying the variables of a predicate, ***one must qualify each variable only once***. For example "$\exists x \in \mathbb{R}, \exists x \in \mathbb{Z}, x = 1$" is not an appropriate statement. Similarly, ***one must not use a single symbol for two different qualified variables***. For example, let $\mathfrak{a}_1 := (\exists x \in \mathbb{R}, 1 < x)$ and $\mathfrak{a}_2 := (\exists x \in \mathbb{R}, x < 0)$ which are true statements respectively obtained by qualifying the predicates $\mathfrak{p}_1(x) := (1 < x)$ and $\mathfrak{p}_2(x) := (x < 0)$. Now, because $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are true, there is a real number x such that $1 < x$ (according to $\mathfrak{a}_1$) and $x < 0$ (according to $\mathfrak{a}_2$). But, would not this imply $1 < 0$? The fallacy of this argument is in our illegitimate use of x for two different purposes, once as a qualified variable in $\mathfrak{a}_1$ and then as a qualified variable in $\mathfrak{a}_2$. To avoid this fallacy, we reserve x for the variable of $\mathfrak{p}_1$ that appears in $\mathfrak{a}_1$ and use y to denote the variable of $\mathfrak{p}_2$ that appears in $\mathfrak{a}_2$, i.e., write $\mathfrak{a}_2 := (\exists y \in \mathbb{R}, y < 0)$. In this way we are allowed to use x and y in other arguments, e.g., to establish the statement: "$\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, y < x$," that follows from $\mathfrak{a}_1$ and $\mathfrak{a}_2$ and the inequalities: $y < 0 < 1 < x$.

Next, consider comparing the statement $\mathfrak{a} :=$ "for every real number x there is an integer n such that $x > n$," i.e., $\mathfrak{a} := (\forall x \in \mathbb{R}, \exists n \in \mathbb{Z},\ x > n)$, with the statement $\mathfrak{b} := (\exists n \in \mathbb{Z}, \forall x \in \mathbb{R},\ x > n)$. It is not difficult to see that $\mathfrak{a}$ is true while $\mathfrak{b}$ is false. Therefore, although $\mathfrak{b}$ is obtained from $\mathfrak{a}$ by changing the order in which "$\forall x \in \mathbb{R}$" and "$\exists n \in \mathbb{Z}$" appear, $\mathfrak{a}$ and $\mathfrak{b}$ are different statements. This example shows that ***changing the position of different terms appearing in a statement may change the statement altogether***.

In mathematical theories whenever one defines a new object, one must address the natural question of its *existence*. For example one may define "m"to be "the greatest natural number." But such a natural number does not exist. ***Defining a mathematical object does not imply its existence.*** The latter must be established independently.

Once one assures that a mathematical object exists, one must enquire into the question of its *uniqueness*. For example, let p and q be a pair of rational numbers satisfying $0 < p < q$ and $p^2 + q^2 = 1$. Such a pair exists because one can produce the example: $p = 3/5$ and $q = 4/5$. But this pair is not the only one. Another example is $p = 5/13$ and $q = 12/13$. The above two examples show that the relations $0 < p < q$ and $p^2 + q^2 = 1$ have a rational solution (p, q) which is however not unique.

The existence and uniqueness problems are of fundamental importance in all areas of mathematics. The qualifier $\exists!$ is often used to imply unique existence. It stands for "***there exists one and only one***." We can express the non-uniqueness discussed in the preceding paragraph as the statement:

$$\text{“}(\exists! p \in \mathbb{Q},\ \exists! q \in \mathbb{Q},\ 0 < p < q,\ p^2 + q^2 = 1) \text{ is false.”}$$

The following is another example of a uniqueness statement.

$$\exists! m \in \mathbb{Z},\ \forall n \in \mathbb{Z},\ m + n = n.$$

2.3 Negation

To each statement we can associate another statement negating it.

Definition 2.3.1 *Let* $\mathfrak{a}$ *be a statement. Then the statement "*$\mathfrak{a}$ *is false."is called the* ***negation*** *of* $\mathfrak{a}$ *and denoted by* $\neg\mathfrak{a}$. ■

If $\mathfrak{a}$ happens to be true, then $\neg\mathfrak{a}$ is false and if $\mathfrak{a}$ is false then $\neg\mathfrak{a}$ is true. This shows that the truth value of $\neg\mathfrak{a}$ is the opposite of that of $\mathfrak{a}$. It is usually convenient to construct a table giving various possibilities for the truth values of unspecified statements. Such a table is called a ***truth table***. A simple example is Table 2.1. Its first column shows the two possible truth values of

$\mathfrak{a}$	$\neg\mathfrak{a}$
T	F
F	T

Table 2.1: Truth table for $\neg$

$\mathfrak{a}$. Its second column gives the corresponding truth values for $\neg\mathfrak{a}$. We may view Table 2.1 as an alternative definition of negation. We can use it to establish the following simple property of negation.

Proposition 2.3.1 *Let* $\mathfrak{a}$ *be a statement. Then* $\neg(\neg\mathfrak{a})$ *is logically equivalent to* $\mathfrak{a}$*, i.e.,* $(\neg(\neg\mathfrak{a})) \Leftrightarrow \mathfrak{a}$.

Proof: It suffices to extend Table 2.1 to include the truth values of $\neg(\neg\mathfrak{a})$. This yields Table 2.2 showing that $\mathfrak{a}$ and $\neg(\neg\mathfrak{a})$ have the same truth value. Hence, according to Definition 2.1.2, they are logically equivalent. ■

We can extend the above definition of negation to predicates.

Definition 2.3.2 *Let* $\mathfrak{p}$ *be a predicate. Then the predicate "*$\mathfrak{p}$ *is false." is called the* ***negation*** *of* $\mathfrak{p}$ *and denoted by* $\neg\mathfrak{p}$. ■

$\mathfrak{a}$	$\neg\mathfrak{a}$	$\neg(\neg\mathfrak{a})$
T	F	T
F	T	F

Table 2.2: Truth table for $\neg\neg$

Clearly, $\neg\mathfrak{p}$ depends on the same variable(s) as $\mathfrak{p}$ does. It is true (respectively false) for those values of the variable(s) for which $\mathfrak{p}$ is false (respectively true).

Next, we consider the problem of negating statements that involve qualifiers $\forall$, $\exists$, and $\exists!$.

Let $\mathfrak{a}$ be the statement: "$\forall n \in \mathbb{Z},\ n = 1$." To negate this statement, we must produce at least one integer that is different from 1. Expressing this in mathematical symbols we have: $\neg\mathfrak{a} = (\exists n \in \mathbb{Z},\ n \neq 1)$.[4] Next, consider the statement $\mathfrak{b} := (\exists r \in \mathbb{Q},\ r^2 + r^{-2} = 1)$. To negate $\mathfrak{b}$ we must show that for every rational number r the equality $r^2 + r^{-2} = 1$ is false. Therefore, $\neg\mathfrak{b} = (\forall r \in \mathbb{Q},\ r^2 + r^{-2} \neq 1)$. A straightforward application of this argument establishes the following theorem.

Theorem 2.3.1 *Let $\mathfrak{p}(x)$ be a predicate whose variables (collectively denoted by x) belong to a set A, and $\mathfrak{c}$ and $\mathfrak{d}$ be the statements:*

$$\mathfrak{c} := (\forall x \in A,\ \mathfrak{p}(x)), \qquad \mathfrak{d} := (\exists x \in A,\ \mathfrak{p}(x)). \tag{2.1}$$

Then

$$\neg\mathfrak{c} = (\exists x \in A,\ \neg\mathfrak{p}(x)), \qquad \neg\mathfrak{d} = (\forall x \in A,\ \neg\mathfrak{p}(x)). \tag{2.2}$$

Proof: Equations (2.2) follow from the same argument that we used to deal with the examples given in the preceding paragraph. ■

Exercise 2.3.1 Find $\neg\mathfrak{a}$ for $\mathfrak{a} := (\forall x \in \mathbb{R}, \exists n \in \mathbb{Z}, x < n)$.
Solution: First we express $\mathfrak{a}$ in one of the forms given in (2.1). We can do this by introducing $\mathfrak{p}(x) := (\exists n \in \mathbb{Z}, x < n)$, so that $\mathfrak{a} = (\forall x \in \mathbb{R}, \mathfrak{p}(x))$. Then in view of (2.2), we have

$$\neg\mathfrak{a} = (\exists x \in \mathbb{R}, \neg\mathfrak{p}(x)). \tag{2.3}$$

Now, we apply (2.2) once again for $\mathfrak{p}(x)$ to find $\neg\mathfrak{p}(x) = (\forall n \in \mathbb{Z}, x \geq n)$. Combining this relation with (2.3), we obtain

$$\neg\mathfrak{a} = (\exists x \in \mathbb{R}, \forall n \in \mathbb{Z}, x \geq n). \quad ■$$

This completes our discussion of negating statements involving $\forall$ and $\exists$. Next, we consider negating an statement involving $\exists!$.

Exercise 2.3.2 Find $\neg\mathfrak{u}$ for $\mathfrak{u} := (\exists! n \in \mathbb{Z}^+, n^2 < 2)$.
Solution: There are two ways in which we can negate $\mathfrak{u}$. Either we must show that there is no positive integer n satisfying $n^2 < 2$ or produce at least two (different) positive integers n_1 and n_2 such that $n_1^2 < 2$ and $n_2^2 < 2$. The first strategy indeed negates $\mathfrak{e} := (\exists n \in \mathbb{Z}^+, n^2 < 2)$ which amounts to $\neg\mathfrak{e} = (\forall n \in \mathbb{Z}^+, n^2 \geq 2)$. So let us assume that $\mathfrak{e}$ is true (as it is), and pursue the second strategy which is to actually negate the uniqueness feature of $\mathfrak{u}$. In mathematical symbols we can express it in the form:

$$\mathfrak{f} := (\exists n_1 \in \mathbb{Z}^+,\ \exists n_2 \in \mathbb{Z}^+,\ n_1 \neq n_2,\ n_1^2 < 2,\ n_2^2 < 2). \tag{2.4}$$

Strictly speaking, $\neg\mathfrak{u}$ asserts that either $\neg\mathfrak{e}$ or $\mathfrak{f}$ is true. This is an example of a compound statement. ■

[4] The reader should be able to justify our choice of using "=" in place of ":=" in the preceding relation and appreciate the fact that we could use another symbol say "m" in the expression for $\neg\mathfrak{a}$ instead of "n."