Math 107: Solutions to Final Exam problems

Problem 1

1.a (5 points) Explain the difference between a sequence of numbers and a series of numbers and give the definition of a convergent series.

A sequence of numbers is a function $s: \mathbb{Z}^+ \to \mathbb{R}$ that can be represented by an ordered infinite list of its values $s(n)$ that we label as s_n, i.e., write s as $\{s_n\}$. A series $\sum_{n=1}^{\infty} s_n$ is a formal sum of the terms of a sequence $\{s_n\}$ that correspond to the sequence of partial sums of $\{s_n\}$, i.e., $\{S_n\}$ where $S_n := s_1 + s_2 + \cdots + s_n$.

$\sum_{n=1}^{\infty} s_n$ is said to be convergent if $\{S_n\}$ converges.

1.b (5 points) Give the definition of the harmonic series, and use the integral test to determine if it converges or not.

Harmonic series $:= \{\frac{1}{n}\}$

Let $f: [1,\infty) \to \mathbb{R}$ be the function $f(x) := \frac{1}{x}$

So that $\frac{1}{n} = f(n)$.

Because $\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{dx}{x} = \lim_{N\to\infty} \ln(x) - \ln(1) = \infty$,

the integral test implies that $\{\frac{1}{n}\}$ diverges.

Problem 2 Let $f(x) := \int_0^x \frac{1-e^{-t^4}}{t^2}\,dt$.

2.a (5 points) Obtain the Maclaurin series for f.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Rightarrow e^{-x^4} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!} = 1 - x^4 + \frac{x^8}{2!} - \frac{x^{12}}{3!} \pm \cdots$$

$$\Rightarrow \frac{1-e^{-t^4}}{t^2} = \frac{1}{t^2}\left[1 - \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n}}{n!}\right] = \frac{1}{t^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} t^{4n}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} t^{4n-2}}{n!}$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \int_0^x t^{4n-2}\,dt = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \left(\frac{t^{4n-1}}{4n-1}\right)\Big|_0^x$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n-1}}{n!\,(4n-1)}$$

$$= \frac{x^3}{3} - \frac{x^7}{(2!)(7)} + \frac{x^{11}}{(3!)(11)} \pm \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(n+1)!\,(4n+3)}$$

2.b (5 points) Give an estimate for the value of $f(1)$ that differs from the exact value by less than 0.01.

$$f(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!\,(4n+3)} = \frac{1}{3} - \frac{1}{(2!)(7)} + \frac{1}{(3!)(11)} - \frac{1}{(4!)(15)} \pm \cdots$$

$$= \frac{1}{3} - \frac{1}{14} + \frac{1}{66} - \frac{1}{24\times 15} \pm \cdots$$

$$\hookrightarrow \frac{1}{360} < \frac{1}{100}$$

$$\Rightarrow f(1) \approx \frac{1}{3} - \frac{1}{14} + \frac{1}{66} = \frac{7\times 22 - 33 + 7}{7\times 66} = \frac{154-26}{7\times 66}$$

$$= \frac{128}{7\times 66} = \frac{64}{7\times 33} = \frac{64}{231}$$

Problem 3 Let V be a complex vector space, A and B be two nonempty subsets of V, and C be a subset of V whose elements are of the form $a+b$ for some $a \in A$ and $b \in B$, i.e.,

$$C := \{a+b \mid a \in A,\ b \in B\}.$$

3.a (5 points) Show that C is a subset of the span of $A \cup B$, i.e., $C \subseteq \langle A \cup B \rangle$.

$\forall c \in C,\ \exists a \in A,\ b \in B,\quad c = a+b$

$\Downarrow \qquad \Downarrow$

$a \in A \cup B \qquad b \in A \cup B$

$A \cup B \subseteq \langle A \cup B \rangle \Longrightarrow a, b \in \langle A \cup B \rangle$

$\langle A \cup B \rangle$ is a subspace of V $\Big\} \Rightarrow a + b \in \langle A \cup B \rangle$

$\Rightarrow \quad c \in \langle A \cup B \rangle \quad \Rightarrow \quad C \subseteq \langle A \cup B \rangle.$

3.b (5 points) Show that if A and B are subspaces of V, then C is also a subspace of V.

1) A & B are subspace $\Rightarrow 0 \in A$ and $0 \in B \Rightarrow$

$0 = 0 + 0 \in C \quad \Rightarrow \quad 0 \in C$

2) $\forall c_1, c_2 \in C \qquad \forall \alpha_1, \alpha_2 \in \mathbb{C}$

$\Downarrow$

$\exists a_1, a_2 \in A,\ \exists b_1, b_2 \in B,\quad c_1 = a_1 + b_1$

$c_2 = a_2 + b_2$

$\Rightarrow \quad \alpha_1 c_1 + \alpha_2 c_2 = \alpha_1 (a_1 + b_1) + \alpha_2 (a_2 + b_2)$

$= \alpha_1 a_1 + \alpha_2 a_2 + \alpha_1 b_1 + \alpha_2 b_2$

A is a subspace, $a_1, a_2 \in A$ $\Big\} \Rightarrow \alpha_1 a_1 + \alpha_2 a_2 \in A$

B is a subspace, $b_1, b_2 \in A$ $\Big\} \Rightarrow \alpha_1 b_1 + \alpha_2 b_2 \in B$

$\alpha_1 c_1 + \alpha_2 c_2 \in C$

① & ② $\Rightarrow C$ is a subspace of V.

Problem 4 Let $\mathfrak{M}(2,2;\mathbb{R})$ be the vector space of 2×2 real matrices and $L : \mathfrak{M}(2,2;\mathbb{R}) \to \mathfrak{M}(2,2;\mathbb{R})$ be the function defined on $\mathfrak{M}(2,2;\mathbb{R})$ according to: $\forall \mathbf{M} \in \mathfrak{M}(2,2;\mathbb{R})$, $L(\mathbf{M}) := \mathbf{M}^T$, where $\mathbf{M}^T$ stands for the transpose of $\mathbf{M}$, i.e., if $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $L(\mathbf{M}) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

4.a (5 points) Show that L is a linear operator.

1) $Dom(L) = \mathfrak{M}(2,2,\mathbb{R})$ which is a subspace of $\mathfrak{M}(2,2,\mathbb{R})$

2) $\forall\ \mathbb{M}_1 := \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $\mathbb{M}_2 := \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in \mathfrak{M}(2,2,\mathbb{R})$

$, \forall\ \alpha_1, \alpha_2 \in \mathbb{R}$

$$L(\alpha_1 \mathbb{M}_1 + \alpha_2 \mathbb{M}_2) = \begin{bmatrix} \alpha_1 a_1 + \alpha_2 a_2 & \alpha_1 b_1 + \alpha_2 b_2 \\ \alpha_1 c_1 + \alpha_2 c_2 & \alpha_1 d_1 + \alpha_2 d_2 \end{bmatrix}^T$$

$$= \begin{bmatrix} \alpha_1 a_1 + \alpha_2 a_2 & \alpha_1 c_1 + \alpha_2 c_2 \\ \alpha_1 b_1 + \alpha_2 b_2 & \alpha_1 d_1 + \alpha_2 d_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} a_1 & c_1 \\ b_1 & d_1 \end{bmatrix} + \alpha_2 \begin{bmatrix} a_2 & c_2 \\ b_2 & d_2 \end{bmatrix}$$

$$= \alpha_1 \mathbb{M}_1^T + \alpha_2 \mathbb{M}_2^T = \alpha_1 L(\mathbb{M}_1) + \alpha_2 L(\mathbb{M}_2)$$

① & ② $\Rightarrow$ L is a linear operator.

4.b (5 points) Determine the null space of L.

$\forall\ \mathbb{M} \in Nul(L)$ $\quad L(\mathbb{M}) = \mathbb{O}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow a = c = b = d = 0$

$\Rightarrow \quad \mathbb{M} = \mathbb{O} \quad \Rightarrow \quad Nul(L) = \{\mathbb{O}\}$.

4.c (5 points) Determine if L is an isomorphism. Justify your response.

Because $Nul(L) = \{\mathbb{O}\}$, L is 1-to-1. $\mathfrak{M}(2,2;\mathbb{R})$ is finite-dimensional and L is everywhere-defined

$\Leftrightarrow$ L is onto $\Leftrightarrow$ L is an isomorphism.

4.d (5 points) Let $B := \{\mathbf{E}^{(1)}, \mathbf{E}^{(2)}, \mathbf{E}^{(3)}, \mathbf{E}^{(4)}\}$ be the standard basis of $\mathfrak{M}(2,2;\mathbb{R})$, i.e.,

$$\mathbf{E}^{(1)} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{E}^{(2)} := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{E}^{(3)} := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{E}^{(4)} := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find the matrix representation of L in B.

$$L(\mathbb{E}^{(1)}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \mathbb{E}^{(1)}$$

$$L(\mathbb{E}^{(2)}) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \mathbb{E}^{(3)}$$

$$L(\mathbb{E}^{(3)}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E^{(2)}$$

$$L(\mathbb{E}^{(4)}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = E^{(4)}$$

$\Downarrow$

The matrix representation of L in B is:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 5 (10 points) Use the method of Gaussian elimination to determine the value(s) of α for which the following system of equations has one or more solutions and find the general form of its solution for this value(s) of α.

$$x - 2i\,y + 3z = \alpha$$

$$x - 8i\,y + 2z = 6$$

$$x + 4i\,y + 4z = 6$$

$$\left[\begin{array}{ccc|c} 1 & -2i & 3 & \alpha \\ 1 & -8i & 2 & 6 \\ 1 & 4i & 4 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2i & 3 & \alpha \\ 0 & -6i & -1 & 6-\alpha \\ 0 & 6i & 1 & 6-\alpha \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -2i & 3 & \alpha \\ 0 & -6i & -1 & 6-\alpha \\ 0 & 0 & 0 & 2(6-\alpha) \end{array}\right]$$

So the system has solutions iff $\boxed{\alpha = 6}$.

In this case we find

$$\left[\begin{array}{ccc|c} 1 & -2i & 3 & 6 \\ 0 & -6i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This corresponds to

$$x - 2iy + 3z = 6$$

$$-6iy - z = 0 \Rightarrow y = \frac{z}{-6i} = \frac{iz}{6}$$

$$0 = 0$$

$$x - 2i\left(\frac{iz}{6}\right) + 3z = 6$$

$$x + \frac{z}{3} + 3z = 6$$

$$\Rightarrow \boxed{x = 6 - \left(3z + \frac{z}{3}\right)}$$

$$\boxed{y = \frac{iz}{6}}$$

z is an arbitrary complex number $\hookrightarrow$ the system has ∞'ly many solutions.

Problem 6 Let $\mathbf{M}$ be an $n \times n$ matrix, $\mathbf{I}$ be the $n \times n$ identity matrix, and λ be a complex number.

6.a (5 points) Show that if λ is an eigenvalue of $\mathbf{M}$, then $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$.

$\Rightarrow \exists\, a \in \mathbb{C}^n,\ a \neq 0,\quad \mathbb{M} a = \lambda a \Rightarrow (\mathbb{M} - \lambda \mathbb{I})\, a = 0$

If $\det(\mathbb{M} - \lambda \mathbb{I}) \neq 0 \Rightarrow \mathbb{M} - \lambda \mathbb{I}$ is invertible

$$\underbrace{(\mathbb{M} - \lambda a)^{-1} (\mathbb{M} - \lambda \mathbb{I})\, a}_{} = (\mathbb{M} - \lambda \mathbb{I})^{-1} 0$$

$a = 0$ which is a contradiction $\Rightarrow \det(\mathbb{M} - \lambda \mathbb{I}) = 0$.

6.b (5 points) Show that if $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$, then λ is an eigenvalue of $\mathbf{M}$.

Let $M : \mathbb{C}^n \to \mathbb{C}^n$ be the linear op defined by $Mx := \mathbb{M}x$. If $\det(\mathbb{M} - \lambda \mathbb{I}) = 0$, $\mathbb{M} - \lambda \mathbb{I}$ does not have an inverse $\Leftrightarrow$ $M - \lambda I : \mathbb{C}^n \to \mathbb{C}^n$ is not invertible

identity operator: $\mathbb{C}^n \to \mathbb{C}^n$

$\Updownarrow$

$\mathrm{Nul}(M - \lambda I) \neq \{0\} \Rightarrow \exists\, a \in \mathbb{C}^n,\ a \neq 0,\ (M - \lambda I) a = 0$

But $(M - \lambda a) a = \mathbb{M} a - \lambda a$

$\mathbb{M} a - \lambda a = 0 \Rightarrow M a = \lambda a$

$\Updownarrow$

λ is an eigenvalue of $\mathbb{M}$.

6.c (5 points) Show that every eigenvalue of $\mathbf{M}$ is also an eigenvalue of the transpose of $\mathbf{M}$.

Let α be an eigenvalue of $\mathbb{M} \Rightarrow \det(\mathbb{M} - \alpha \mathbb{I}) = 0$

$\Rightarrow \det\left((\mathbb{M} - \alpha \mathbb{I})^T\right) = 0 \Rightarrow \det(\mathbb{M}^T - \alpha \mathbb{I}^T) = 0$

$\Rightarrow \det(\mathbb{M}^T - \alpha \mathbb{I}) = 0 \Rightarrow \alpha$ is an eigenvalue of $\mathbb{M}^T$.

Problem 7 (10 points) Let $L : V \to V$ be a linear operator acting in V and having three distinct eigenvalues λ_1, λ_2, and λ_3. Suppose that v_1, v_2, and v_3 are eigenvectors of L with eigenvalues λ_1, λ_2, and λ_3, respectively. Prove that $\{v_1, v_2, v_3\}$ is a linearly-independent subset of V.

$$Lv_1 = \lambda_1 v_1, \quad Lv_2 = \lambda_2 v_2, \quad Lv_3 = \lambda_3 v_3$$

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \quad (I)$$

Apply $L - \lambda_1 I \Rightarrow (L-\lambda_1 I)(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3) = 0$

$$\Rightarrow (L-\lambda_1 I)\alpha_1 v_1 + (L-\lambda_1 I)\alpha_2 v_2 + (L-\lambda_1 I)\alpha_3 v_3 = 0$$

$$\Rightarrow \alpha_1 \underbrace{(L-\lambda_1 I)v_1}_{\lambda_1 v_1 - \lambda_1 v_1 = 0} + \alpha_2 \underbrace{(L-\lambda_1 I)v_2}_{\lambda_2 v_2 - \lambda_1 v_2 = (\lambda_2-\lambda_1)v_2} + \alpha_3 \underbrace{(L-\lambda_1 I)v_3}_{\lambda_3 v_3 - \lambda_1 v_3 = (\lambda_3-\lambda_1)v_3} = 0$$

$$\Rightarrow \alpha_2(\lambda_2-\lambda_1)v_2 + \alpha_3(\lambda_3-\lambda_1)v_3 = 0 \quad (II)$$

Apply $L - \lambda_2 I \Rightarrow (L-\lambda_2 I)\left[\alpha_2(\lambda_2-\lambda_1)v_2 + \alpha_3(\lambda_3-\lambda_1)v_3\right] = 0$

$$\Rightarrow \alpha_2(\lambda_2-\lambda_1)\underbrace{(L-\lambda_2 I)v_2}_{\lambda_2 v_2 - \lambda_2 v_2 = 0} + \alpha_3(\lambda_3-\lambda_1)\underbrace{(L-\lambda_2 I)v_3}_{\lambda_3 v_3 - \lambda_2 v_3 = (\lambda_3-\lambda_2)v_3} = 0$$

$$\Rightarrow \alpha_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)v_3 = 0$$

$v_3 \neq 0$, $\lambda_3-\lambda_1 \neq 0$, $\lambda_3-\lambda_2 \neq 0 \Rightarrow \boxed{\alpha_3 = 0}$ (III)

II & III $\Rightarrow \alpha_2(\lambda_2-\lambda_1)v_2 = 0$
$(\lambda_2-\lambda_1) \neq 0$, $v_2 \neq 0 \Rightarrow \boxed{\alpha_2 = 0}$ (IV)

I & III & IV $\Rightarrow \alpha_1 v_1 = 0$
$v_1 \neq 0 \Rightarrow \boxed{\alpha_1 = 0}$

So $\{v_1, v_2, v_3\}$ is linearly-independent.

Problem 8 Let $\mathbf{M} := \begin{bmatrix} 1 & -\alpha \\ 0 & 2+\alpha \end{bmatrix}$ where α is a real parameter.

8.a (10 points) Solve the eigenvalue problem for $\mathbf{M}$, i.e., find its eigenvalues and the general form of the corresponding eigenvectors.

$$\mathbb{A}_\lambda := \begin{bmatrix} 1-\lambda & -\alpha \\ 0 & 2+\alpha-\lambda \end{bmatrix} \qquad \det \mathbb{A}_\lambda = 0$$

$$(1-\lambda)(2+\alpha-\lambda) = 0 \Rightarrow \begin{cases} \lambda = 1 \\ \lambda = 2+\alpha \end{cases}$$

For $\lambda := \lambda_1 = 1$, $\quad \mathbb{A}_\lambda \mathbf{a}_1 = \mathbf{0} \Rightarrow \begin{bmatrix} 0 & -\alpha \\ 0 & 1+\alpha \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}}_{\mathbf{a}_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{array}{l} -\alpha\beta_1 = 0 \\ (1+\alpha)\beta_1 = 0 \end{array} \Rightarrow \boxed{\beta_1 = 0}$$

$$\Rightarrow \mathbf{a}_1 = \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $\lambda := \lambda_2 = 2+\alpha$, $\quad \mathbb{A}_\lambda \mathbf{a}_2 = \mathbf{0} \Rightarrow \begin{bmatrix} -1-\alpha & -\alpha \\ 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}}_{\mathbf{a}_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow -(1+\alpha)\alpha_2 - \alpha\beta_2 = 0$$

$$\Rightarrow \alpha\beta_2 = -(1+\alpha)\alpha_2$$

If $\alpha \neq 0$ $\Rightarrow$ $\beta_2 = -(1+\frac{1}{\alpha})\alpha_2 \Rightarrow \mathbf{a}_2 = \begin{bmatrix} \alpha_2 \\ -(1+\frac{1}{\alpha})\alpha_2 \end{bmatrix} = \alpha_2 \begin{bmatrix} 1 \\ -(1+\frac{1}{\alpha}) \end{bmatrix}$

If $\alpha = 0$ $\Rightarrow$ $\alpha_2 = 0 \Rightarrow \mathbf{a}_2 = \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} = \beta_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

8.b (05 points) Find all values of α such that $\mathbf{M}$ is not diagonalizable.

For $\mathbb{M}$ to be non-diagonalizable $\mathbf{a}_1$ & $\mathbf{a}_2$ must be proportional so that there is not basis of $\mathbb{C}^2$ consisting of the eigenvectors of $\mathbb{M}$.

This does not happen for $\alpha = 0$.

It happens for $\alpha \neq 0$ if

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \gamma\, \alpha_2 \begin{bmatrix} 1 \\ -(1+\frac{1}{\alpha}) \end{bmatrix} \Rightarrow 1 + \frac{1}{\alpha} = 0$$

$$\Downarrow$$

$$\boxed{\alpha = -1}$$