Math 107, Fall 2012, Quiz # 3a

You have 30 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points) Give the definition of the following terms.

1.a) Span of a subset of a vector space: Let S be a nonempty subset of a vector space V. Then span of S is the set of all linear combinations of the elements of S.

1.b) Basis of a vector space: Basis of a vector space V is a subset of V which is linearly independent and spans V.

1.c) Finite dimensional vector space:
A vector space V is called a finite dimensional vector space if either $V=\{0\}$ or V has a finite basis.

Problem 2 (5 points) Find polar representation of i, $-i$, $\sqrt{3}+i$.

$$\left.\begin{array}{l} |i|=1 \\ \operatorname{Arg} i = \frac{\pi}{2} \end{array}\right\} \quad i = e^{i\pi/2}$$

$$\left.\begin{array}{l} |-i|=1 \\ \operatorname{Arg}(-i) = \frac{3\pi}{2} \end{array}\right\} \quad -i = e^{\frac{i3\pi}{2}}$$

$$\left.\begin{array}{l} |\sqrt{3}+i| = \sqrt{3+1} = 2 \\ \operatorname{Arg}|\sqrt{3}+i| = \frac{\pi}{6} \end{array}\right\} \quad \sqrt{3}+i = 2e^{\frac{i\pi}{6}}$$

Problem 3 (12 points) Let $\mathcal{P}_2(\mathbb{R}, \mathbb{R})$ be the vector space of polynomials $p : \mathbb{R} \to \mathbb{R}$ of degree not greater than 2, and $q_1, q_2, q_3, q_4 \in \mathcal{P}_2(\mathbb{R}, \mathbb{R})$ be defined by $q_1(x) := 1 - x$, $q_2(x) := x - x^2$, $q_3(x) := x^2 + 1$, $q_4(x) := 1 + 2x + 3x^2$.

3.a) Show that $\mathcal{B} := \{q_1, q_2, q_3\}$ spans the real vector space $\mathcal{P}_2(\mathbb{R}, \mathbb{R})$.

A general element of $P_2(\mathbb{R},\mathbb{R})$ is of the form $q(x) := ax^2 + bx + d$ where $a,b,d \in \mathbb{R}$, $x \in \mathbb{R}$

We should find $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ s.t.

$$ax^2 + bx + d = \alpha_1(1-x) + \alpha_2(x - x^2) + \alpha_3(x^2+1)$$

$$\Rightarrow \left.\begin{array}{l} \alpha_1 + \alpha_3 = d \\ -\alpha_1 + \alpha_2 = b \\ -\alpha_2 + \alpha_3 = a \end{array}\right\} \quad \left.\begin{array}{l} \alpha_3 = \frac{a+d+b}{2} \\ \alpha_1 = \frac{d-a-b}{2} \\ \alpha_2 = \frac{d+b-a}{2} \end{array}\right\}$$

$$\Rightarrow ax^2 + bx + d = \left(\frac{d-a-b}{2}\right)(1-x) + \left(\frac{d+b-a}{2}\right)(x - x^2) + \left(\frac{a+d+b}{2}\right)(x^2+1) \quad \forall x \in \mathbb{R}.$$

3.b) Show that $\mathcal{B} := \{q_1, q_2, q_3\}$ is a linearly - independent subset of $\mathcal{P}_2(\mathbb{R}, \mathbb{R})$.

Assume $c_1 q_1 + c_2 q_2 + c_3 q_3 = \bar{0}$ → zero polynomial

$$\Rightarrow c_1 q_1(x) + c_2 q_2(x) + c_3 q_3(x) = 0 \in \mathbb{R} \quad \forall x \in \mathbb{R},$$

$$\Rightarrow c_1(1-x) + c_2(x - x^2) + c_3(x^2+1) = 0$$

$$\Rightarrow c_1 = c_2 = c_3 = 0$$

3.c) Find components of q_4 in $\mathcal{B}$ and write the representation of q_4 in $\mathcal{B}$.

$$q_4(x) = 3x^2 + 2x + 1 \underset{\substack{\text{using} \\ \text{part (a)}}}{=} -2\cdot(1-x) + 0\cdot(x - x^2) + 3(x^2+1), \quad \forall x \in \mathbb{R}$$

$$\Rightarrow q_4 = -2q_1 + 3q_3$$

Math 107, Fall 2012, Quiz # 3b

You have 30 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points) Give the definition of the following terms.

1.a) Determine the truth value of the following statements.(T:= true statement, F:= false statement)

(i) $\forall \alpha \in \mathbb{Z}^+$, $\forall \beta \in \mathbb{Q}$, $\alpha < \beta$. False. Since $1 \in \mathbb{Z}^+$, $1/2 \in \mathbb{Q}$ and $\frac{1}{2} \not< 1$

(ii) $(T \Rightarrow F) \vee (F \Rightarrow T)$.

$\equiv F \vee T \equiv T$ (True)

1.b) Write the negation of the following statement.

$\forall \alpha \in \mathbb{Q}$, $\forall \beta \in \mathbb{R}$, $\alpha^2 \leq \beta + \alpha$.

$\exists \alpha \in \mathbb{Q}$, $\exists \beta \in \mathbb{R}$ s.t. $\alpha^2 > \beta + \alpha$

Problem 2 (5 points) Show that

$$V := \{(x, y) \in \mathbb{R}^2 | x + y = 0\}$$

is a subspace of $\mathbb{R}^2$.

(i) V is nonempty, since $(1,-1) \in V$ (nonempty subset)

(ii) Let $a := (a_1, a_2)$, $b := (b_1, b_2) \in V$, Then $a_1 + a_2 = 0$, $b_1 + b_2 = 0$

which implies $a + b = (a_1 + b_1, a_2 + b_2) \in V$

since $a_1 + b_1 + a_2 + b_2 = a_1 + a_2 + b_1 + b_2 = 0$ (closed under addition)

(iii) Let $\alpha \in \mathbb{R}$, $a := (a_1, a_2) \in V \Rightarrow \alpha a = (\alpha a_1, \alpha a_2)$

$\alpha a_1 + \alpha a_2 = \alpha(a_1 + a_2) = 0$

Since $a \in V$. (closed under scalar multiplication)

Problem 3 (12 points) Let $\vec{a}_1 := (1, i)$ and $\vec{a_2} := (i, 1)$.

3.a) Show that $\{\vec{a}_1, \vec{a}_2\}$ spans the complex vector space $\mathbb{C}^2$.

$\{\vec{a}_1, \vec{a}_2\}$ spans $\mathbb{C}^2$ means arbitrary element $(w,z) \in \mathbb{C}^2$ can be written as a linear combination of elements of $\{\vec{a}_1, \vec{a}_2\}$

In our case:

$$(w,z) = \frac{w-iz}{2}(1,i) + \frac{z-iw}{2}(i,1) \Rightarrow \{\vec{a}_1, \vec{a}_2\} \text{ spans } \mathbb{C}^2.$$

(Note: How to find coefficients?

For $\alpha_1, \alpha_2 \in \mathbb{C}$, assume $(w,z) = \alpha_1(1,i) + \alpha_2(i,1)$)

$$\Rightarrow \left.\begin{array}{l} w = \alpha_1 + i\alpha_2 \\ z = i\alpha_1 + \alpha_2 \end{array}\right\} \Rightarrow \begin{array}{l} \alpha_1 = \frac{w-iz}{2} \\ \alpha_2 = \frac{z-iw}{2} \end{array}$$

3.b) Show that $\{\vec{a}_1, \vec{a}_2\}$ is a linearly - independent subset of $\mathbb{C}^2$.

For $c_1, c_2 \in \mathbb{C}$,

If $c_1(1,i) + c_2(i,1) = \vec{0} := (0,0)$

Then $(c_1 + ic_2, ic_1 + c_2) = (0,0)$

$\Rightarrow c_1 = c_2 = 0$

Hence, $\{a_1, a_2\}$ is a linearly independent subset of $\mathbb{C}^2$.

3.c) Determine the coefficients of expansion of $(2 - i, 2i - 1)$ in $\{\vec{a}_1, \vec{a}_2\}$.

Since we know that $(w,z) = \frac{(w-iz)}{2}(1,i) + \frac{(z-iw)}{2}(i,1)$ by part (a)

$$(2-i, 2i-1) = 2(1,i) - (i,1)$$

Math 107, Fall 2012, Quiz # 3c

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points)

1.a) Subspace of a vector space: A subspace of a vector space V is a nonempty subset of V s.t. it is closed under addition and scalar multiplication defined on the vector space V.

1.b) Basis of a vector space: Basis of a vector space V is a subset of V which is linearly independent and spans V.

1.c) Dimension of a a finite dimensional vector space:

If the vector space is trivial ($V = \{0\}$), then dimension is zero. Otherwise the dimension of the vector space is the number of elements of a basis of the vector space.

Problem 2 (5 points) Show that $\mathbb{R}$ is not a subspace of the complex vector space $\mathbb{C}$.

$\mathbb{R}$ is not a subspace of $\mathbb{C}$, since it is not closed under scalar multiplication by a complex number:

e.g.s $2 \in \mathbb{R}$, $i \in \mathbb{C}$, however $2i \notin \mathbb{R}$.

Problem 3 (12 points) Let $\vec{a_1} := (1,1)$ and $\vec{a_2} := (1,-2)$.

3.a) Show that $\mathcal{B} := \{\vec{a_1}, \vec{a_2}\}$ is a basis of the complex vector space $\mathbb{R}^2$.

We show that:

- B spans $\mathbb{R}^2$.

We take an arbitrary elt $(a,b) \in \mathbb{R}^2$

$$(a,b) = \alpha_1(1,1) + \alpha_2(1,-2) \quad \text{for } \alpha_1, \alpha_2 \in \mathbb{R}$$

$$\Rightarrow \left.\begin{array}{l} a = \alpha_1 + \alpha_2 \\ b = \alpha_1 - 2\alpha_2 \end{array}\right\} \Rightarrow \begin{array}{l} \alpha_2 = \frac{a-b}{3} \\ \alpha_1 = \frac{2a+b}{3} \end{array}$$

Hence, for an arbitrary elt. $(a,b) \in \mathbb{R}^2$ $(a,b) = \frac{a-b}{3}(1,1) + \left(\frac{2a+b}{3}\right)(1,-2)$ which means that any elt can be written as a linear combination of $\{a_1, a_2\}$

- B is linearly independent

Assume, $c_1\vec{a_1} + c_2\vec{a_2} = \vec{0} \Rightarrow (c_1+c_2, c_1-2c_2) = (0,0)$

$\Rightarrow c_1 = c_2 = 0$

Hence B is linearly ind.

3.b) Find the components of $(1,2)$ in the basis $\mathcal{B}$ and determine the representation of $(1,2)$ in $\mathcal{B}$

- $(1,2) \underset{\text{by part (a)}}{=} \frac{4}{3}(1,1) - \frac{1}{3}(1,-2)$

- Representation of $(1,2)$ in B = is $\begin{bmatrix} 4/3 \\ -\frac{1}{3} \end{bmatrix}$

Math 107, Fall 2012, Quiz # 3d

You have 30 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points) Give the definition of the following terms.

1.a) Finite dimensional vector space:

A vector space V is called finite dimensional vector space if either $V=\{0\}$ or V has a finite basis.

1.b) Dimension of a finite dimensional vector space:

If $V=\{0\}$ dimension of V is said to be zero.
Otherwise the dimension of the vector space is the number of elements of its basis.

1.c) Subspace of a vector space:

A subspace of a vector space V is a nonempty subset of V s.t. it is closed under addition and scalar mult. defined on the vector space V.

Problem 2 (5 points) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x) := x|x|$. Determine whether f is one-to-one and onto.

- f is one-to-one, because

For $x_1,x_2\in\mathbb{R}$ $f(x_1)=f(x_2) \Rightarrow x_1|x_1|=x_2|x_2|$

$\Rightarrow x_1=x_2$, because $\rightarrow$

Note that
If $x_1<0$ $x_1|x_1|=-x_1^2<0$
$\Rightarrow x_2<0 \Rightarrow x_2|x_2|=-x_2^2$
$\Rightarrow x_1=x_2$, $x_1<0$, $x_2<0$
Similarly
If $x_1>0$ $x_1|x_1|=x_1^2>0$
$\Rightarrow x_2>0$ $x_2|x_2|=x_2^2$
$x_1=x_2$ $x_1>0$, $x_2>0$

and $x_1|x_1|=x_2|x_2|$
$0=0$
$\Rightarrow x_1,x_2=0$

- f is onto since $\mathrm{Dom} f=\mathbb{R}$ and

For $a<0 \Rightarrow -a>0 \Rightarrow a=-\sqrt{-a}\,|-\sqrt{-a}|$, $-\sqrt{-a}\in \mathrm{Dom} f$

For $a=0 \Rightarrow a=0|0|$

For $a>0 \Rightarrow a=\sqrt{a}\,|\sqrt{a}|$, $\sqrt{a}\in\mathrm{Dom} f$

Problem 3 (12 points) Let $\mathcal{P}(\mathbb{R}, \mathbb{R})$ be the vector space of all polynomials $p : \mathbb{R} \to \mathbb{R}$ and $q_1, q_2 \in \mathcal{P}(\mathbb{R}, \mathbb{R})$ be defined by $q_1(x) := x^4 + x^2 + 1, q_2(x) := 2x^4 - 5x^2$.

3.a) (5 points) Show that $\{q_1, q_2\}$ is a linearly independent subset of $\mathcal{P}(\mathbb{R}, \mathbb{R})$.

Assume $c_1 q_1 + c_2 q_2 = 0 \rightarrow$ zero polynomial

$\Rightarrow c_1 q_1(x) + c_2 q_2(x) = 0 \in \mathbb{R} \;, \forall x \in \mathbb{R}$

$c_1 (x^4 + x^2 + 1) + c_2 (2x^4 - 5x^2) = 0$

$\Rightarrow c_1 + 2c_2 = 0$

$c_1 - 5c_2 = 0 \;, c_1 = 0$

$\Rightarrow c_1 = 0 \;, c_2 = 0 \Rightarrow \{q_1, q_2\}$ is linearly ind.

3.b) (4 points) Determine the coefficients of expansion of $r(x) := 5x^4 - 2x^2 + 3$ in $\{q_1, q_2\}$.

For $\alpha_1, \alpha_2 \in \mathbb{R}$, $5x^4 - 2x^2 + 3 = \alpha_1(x^4 + x^2 + 1) + \alpha_2(2x^4 - 5x^2)$

$\Rightarrow \alpha_1 + 2\alpha_2 = 5$

$\alpha_1 - 5\alpha_2 = -2$

$\alpha_1 = 3 \Rightarrow \alpha_2 = 1$

$\Rightarrow 5x^4 - 2x^4 + 3 = 3(x^4 + x^2 + 1) + 1(2x^4 - 5x^2)$

3.c) (3 points) Is $\{q_1, q_2\}$ is a basis for $\mathcal{P}(\mathbb{R}, \mathbb{R})$. Explain why?

No. Because $\{q_1, q_2\}$ can not span $P(\mathbb{R}, \mathbb{R})$

$q(x) := x^5$ can not be written as a linear combination

$q \in P(\mathbb{R}, \mathbb{R})$ of polynomials q_1, q_2 which are of order 4.

Math 107, Fall 2012, Quiz # 3e

You have 30 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points)

1.a) Determine the truth value of the following statements.(T:= true statement, F:= false statement)

(i) $\forall \alpha \in \mathbb{Z}^+,\ \exists \beta \in \mathbb{Q},\ \alpha < \beta.$ True.

(ii) $\underbrace{\underbrace{(T \Rightarrow F)}_{F} \Rightarrow T}_{F \Rightarrow T \text{ is true.} (T)}.$

1.b) Write the negation of the following statement.

$\exists \alpha \in \mathbb{Q},\ \exists \beta \in \mathbb{R},\ \alpha^2 > \beta + \alpha.$

$\forall \alpha \in \mathbb{Q},\ \forall \beta \in \mathbb{R},\ \alpha^2 \leq \beta + \alpha$

Problem 2 (5 points) Show that $V := \{(x, y) \in \mathbb{R}^2 | x + y = 1\}$ is not a subspace of $\mathbb{R}^2$.

V is not a subspace of $\mathbb{R}$. Because, $(1,0) \in V$ but the scalar multiplication of $(1,0)$ by 2 is $(2,0) \notin V$. (not closed under scalar mult.)

Problem 3 (12 points) Let $\vec{a_1} := (1, i)$ and $\vec{a_2} := (i, 1)$.

3.a) (8 points) Show that $B := \{\vec{a}_1, \vec{a}_2\}$ is a basis of the complex vector space $\mathbb{C}^2$.

We show that

- $\{\vec{a}_1, \vec{a}_2\}$ spans $\mathbb{C}^2$. (See Quiz 3b, problem 3a)
- $\{a_1, a_2\}$ is a linearly independent subset of $\mathbb{C}^2$. (See Quiz 3b, problem 3b)

Therefore B is a basis of $\mathbb{C}^2$.

3.b) (4 Points) Find the components of $(1, 0)$ and $(0, 1)$ in the basis B.

Since $(w,z) = \frac{(w-iz)}{2}(1,i) + \frac{(z-iw)}{2}(i,1)$ for any $(w,z) \in \mathbb{C}^2$ (by part (a))

$$(1,0) = \frac{1}{2}(1,i) - \frac{i}{2}(i,1)$$

$$(0,1) = \frac{-i}{2}(1,i) + \frac{1}{2}(i,1)$$

Math 107, Fall 2012, Quiz # 3f

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points) Give the definition of the following terms.

1.a) Span of a subset of a vector space: Let S be a nonempty subset of a vector space V. Then span of S is the set of all linear combinations of the elements of S.

1.b) Linearly independent subset of a vector space:

Let S be a nonempty subset of a vector space V. Then S is linearly independent if $\forall s \in S$, $s \notin \langle S \setminus \{s\} \rangle$.

1.c) Dimension of a finite dimensional vector space:

See Quiz 3d - Problem (1c)

Problem 2 (5 points) Show that

$$V := \{(x, y) \in \mathbb{R}^2 | x + y = 1\}$$

is not a subspace of $\mathbb{R}^2$.

V is not closed under scalar multiplication.

$(1,0) \in V$ however $2(1,0) \notin V$.

since $2+0=2$.

Problem 3 (12 points) Let $\vec{a_1} : (1, i)$ and $\vec{a_2} : (i, 1)$.

3.a) Show that $\mathcal{B} := \{\vec{a_1}, \vec{a_2}\}$ is a basis of the complex vector space $\mathbb{C}^2$.

See Quiz 3e, Problem 3

3.b) Find the components of $(1, 0)$ and $(0, 1)$ in the basis $\mathcal{B}$.

See Quiz 3e, Problem 3.