Math 107, Fall 2012, Quiz # 4a

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points) Give the definition of the following terms.

1.a) Image of a subset under a function

Quiz 4e/1.d

1.b) Null space of a linear operator:

Quiz 4e/1.b

1.c) Range of a function:

Quiz 4d/1.c

Problem 2 (8 points) Let $L : \mathbb{C}^2 \to \mathbb{C}^2$ be an everywhere-defined function given by $L(w, z) := (w - iz, w + iz)$ and $U := \{(0, iz) | z \in \mathbb{C}\}$. Find a basis for $L(U)$ and $L^{-1}(U)$.

i) $L(U) = \{(z, -z)\} \overset{\text{Why?}}{=} \text{span} \underbrace{\{(1,-1)\}}$

This is a linearly independent subset spanning $L(U)$. Hence $\{(1,-1)\}$ is a basis for $L(U)$.

ii) $L^{-1}(U)$

Note that $(w,z) \in L^{-1}(U)$ if and only if $L((w,z)) = (w-iz, w+iz) \in U$ which holds if and only if

$w - iz = 0$

$w + iz = iv$

for some $v \in \mathbb{C}$. (Why can't we write z instead of v? Also we can write v insted of iv. Why?)

Then $w = iz$ from the first equation. Inserting this into the second one we get $2iz = iv$, i.e. $z = \frac{v}{2}$.

Since v is arbitrary z can take any complex value. Then $L^{-1}(U) = \{(iz, z) ; z \in \mathbb{C}\}$. Hence $\{(i, 1)\}$ is a [illegible] for [illegible]

Problem 3 (9 points) Let $\mathcal{P}(\mathbb{R},\mathbb{R})$ be the vector space of real polynomials. Determine whether the functions $L : \mathcal{P}(\mathbb{R},\mathbb{R}) \to \mathcal{P}(\mathbb{R},\mathbb{R})$ are linear operators or not. Here $p \in \mathcal{P}(\mathbb{R},\mathbb{R})$ and $x \in \mathbb{R}$. Justify your response.

Note: As a notation when L is an operator we can write $L(p)$ as Lp.

3.a) $(Lp)(x) := \int_0^x p(t)dt$

i) We will show $L(p+q) = L(p) + L(q)$ for $p, q \in P(\mathbb{R},\mathbb{R})$.

Since both sides of the equality are polynomials (in particular, functions) we have to show, that for each $x \in \mathbb{R}$, they are the same value.

So, let $x \in \mathbb{R}$. Then

$$(L(p+v))(x) = \int_0^x (p(t)+q(t))dt = \int_0^x p(t)dt + \int_0^x q(t)dt$$

$$= (L(p))(x) + (L(q))(x)$$

ii) Now let $\alpha \in \mathbb{R}$. Then

$$(L(\alpha p))(x) = \int_0^x \alpha p(t)dt$$

$$= \alpha \int_0^x p(t)dt$$

$$= \alpha (L(p))(x)$$

Hence, the operator is linear

3.b) $(Lp)(x) := x^2 \int_0^1 p(x)dx$

Let $p, q \in P(\mathbb{R},\mathbb{R})$ and $\alpha \in \mathbb{R}$.

i) $$(L(p+q))(x) = x^2 \int_0^1 (p(x)+q(x))dx$$

$$= x^2 \left(\int_0^1 p(x)dx + \int_0^1 q(x)dx \right)$$

$$= (L(p))(x) + (L(q))(x)$$

ii) $$(L(\alpha p))(x) = x^2 \int_0^1 \alpha p(x)dx$$

$$= \alpha (L(p))(x)$$

3.c) $(Lp)(x) := 2x - 1$

Note that this operator takes every $p \in P(\mathbb{R},\mathbb{R})$ to the same polynomial $q \in P(\mathbb{R},\mathbb{R})$ which is defined as

$$q(x) = 2x - 1.$$

If p is the zero polynomial, then $L(0) = q$ which is not zero. So L is not a linear operator. Why?

Math 107, Fall 2012, Quiz # 4b

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (4 points) Give the definition of the following terms.

1.a) Null space of a ~~function~~: linear operator:

Quiz 4e/1.b

1.b) Image of a subset under a function:

Quiz 4e/1.d

1.c) Inverse image of a subset under a function:

Quiz 4e/1.c

1.d) Range of a function:

Quiz 4d/1.c

Problem 2 (16 points) Let $L : \mathcal{P}_4(\mathbb{R},\mathbb{R}) \to \mathcal{P}_4(\mathbb{R},\mathbb{R})$ be given by $(Lp)(x) := (x^2+1)p(x)$. Here $\mathcal{P}_4(\mathbb{R},\mathbb{R})$ is the real vector space of polynomials from $\mathbb{R}$ to $\mathbb{R}$ of degree at most 4, $p \in \mathcal{P}_4(\mathbb{R},\mathbb{R})$ and $x \in \mathbb{R}$. (*Note that L is not everywhere-defined.*)

2.a) Determine *Dom*(*L*).

A polynomial $p \in P_4(\mathbb{R},\mathbb{R})$ is an element of $Dom(L)$

iff and only if $L(p) \in P_4(\mathbb{R},\mathbb{R})$.

Note that $\deg (x^2+1)p(x) = \deg p + 2$.

So $L(p) \in P_4(\mathbb{R},\mathbb{R})$ iff $\deg p + 2 \leq 4$ iff $\deg p \leq 2$

iff $p \in P_2(\mathbb{R},\mathbb{R})$.

So $Dom(L) = P_2(\mathbb{R},\mathbb{R})$

2.b) Find a basis for *Nul*(*L*).

Suppose that $L(p) \in Nul(L)$.

Then for all $x \in \mathbb{R}$

$(L(p))(x) = (x^2+1)p(x) = 0$ and this implies that $p(x)=0$

for all $x \in \mathbb{R}$. Hence p is the zero polynomial.

Hence $Nul(L) = \{ \underset{\text{zero polynomial}}{0} \}$

So, there is no basis for $Nul(L)$.

2.c) Find a basis for $Ran(L)$.

Note that since $Nul(L) = \{0\}$, L is injective.

Since $\{1, x, x^2\} \subset P_2(\mathbb{R}, \mathbb{R})$ is a basis for $P_2(\mathbb{R}, \mathbb{R})$, $L(\{1, x, x^2\})$ is a basis for $Ran(L)$.

Find the theorems in the book related to this.

Hence $\{L(1), L(x), L(x^2)\} = \{x^2+1, x^3+x, x^4+x^2\}$ is a basis for $Ran(L)$.

2.d) Determine $\dim(DomL)$.

$\dim(Dom\, L) = 3$ since there is a basis consisting of 3 elements.

Exercise: Show that $\{1, x, x^2\}$ is a basis for $Dom(L) = P_2(\mathbb{R}, \mathbb{R})$.

Math 107, Fall 2012, Quiz # 4c

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points) Give the definition of the following terms.

1.a) Image of a subset under a function

Quiz 4e/1.d

1.b) Null space of a linear operator:

Quiz 4e/1.b

1.c) Range of a function:

Quiz 4d/1.c

Problem 2 (5 points) Let $\mathcal{P}(\mathbb{R},\mathbb{R})$ be the vector space of real polynomials. Show that the function $L : \mathcal{P}(\mathbb{R},\mathbb{R}) \to \mathcal{P}(\mathbb{R},\mathbb{R})$ defined by $Lp(x) := \int_0^x p(t)dt$ is a linear operator. Here $p \in \mathcal{P}(\mathbb{R},\mathbb{R})$ and $x \in \mathbb{R}$.

Quiz 4a/3.a

Problem 3 (12 points) Let $L : \mathbb{C}^2 \to \mathbb{C}^2$ be an everywhere-defined function given by $L(w, z) := (w - iz, w + iz)$ for all $(w, z) \in \mathbb{C}^2$.

2.a) (4 points) Determine a basis for the null space of L.

$\text{Nul}(L) = \{(w, z) \in \mathbb{C}^2 : L(w,z) = 0\}$

$L(w,z) = 0 \Rightarrow \left.\begin{array}{l} w - iz = 0 \\ w + iz = 0 \end{array}\right\} \begin{array}{l} w = 0 \\ z = 0 \end{array}$

So $\text{Nul}(L) = \{(0,0)\}$

~~Hence~~ Hence there is no basis for $\text{Nul}(L)$

2.b) (4 points) Determine a basis for the range of L.

Note that L is everywhere-defined, i.e. $\text{Dom}(L) = \mathbb{C}^2$.

We know that

$\text{Dim}(\text{Dom}(L)) = \dim \text{Nul}(L) + \dim \text{Ran}(L)$

$2 = 0 + \dim \text{Ran}(L)$

$\Rightarrow \dim \text{Ran}(L) = 2 \Rightarrow \text{Ran}(L) = \mathbb{C}^2$

why?

$\Downarrow$

$\{(1,0), (0,1)\}$ is a basis for $\mathbb{C}^2$.

2.c) (4 points) Determine if L is one-to-one or onto.

$\text{Nul}(L) = 0 \Rightarrow L$ is one-to-one

$\downarrow$

find this theorem in the book.

Also from part b we have $\text{Ran}(L) = \mathbb{R}^2$.

Hence L is one-to-one and onto.

Math 107, Fall 2012, Quiz # 4d

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points) Give the definition of the following terms.

1.a) A linear operator:

Quiz 4e/1.a

1.b) Null space of a linear operator:

Quiz 4e/1.b

1.c) Range of a function:

Let $f: A \to B$ be a function. Then range of f, denoted by $Ran(f)$ is equal to the set $\{ b \in B : \exists\, a \in A \text{ with } f(a) = b \}$

Problem 1 (5 points) Determine whether the everywhere-defined function $L : \mathbb{C} \to \mathbb{C}$ defined by $L(z) := \bar{z}$ is a linear operator. Justify your response.

Let $\alpha, w, z \in \mathbb{C}$.

(i) $L(\underbrace{\overbrace{w + z}^{\text{vector}}}_{\text{addition of vectors}}) = \overline{w+z} = \bar{w} + \bar{z} = L(w) + L(z)$

(ii) $L(\alpha \cdot w) = \overline{\alpha \cdot w} = \bar{\alpha} \cdot \bar{w} = \bar{\alpha} \cdot L(w)$

scalar

vector

scalar multiplication

In general $\alpha \neq \bar{\alpha}$ for $\alpha \in \mathbb{C}$. Hence L is <u>not</u> a linear operator.

Fact: Let $L: V \to W$ be a linear operator and $\mathcal{B} \subset V$ is a basis for V. Then $\operatorname{span}(L(\mathcal{B})) = \operatorname{Ran}(L)$

Proof: Exercise.

Problem 3 (12 points) Let $L : \mathbb{C}^2 \to \mathbb{C}^2$ be an everywhere-defined linear operator given by $L(w, z) := ((1-i)w - iz, 2w + (1-i)z)$.

2.a) (4 points) Determine a basis for the null space of L.

$\operatorname{Nul}(L) = \{(w,z) : L((w,z)) = (0,0)\}$

So solutions of the following system gives $\operatorname{Nul}(L)$:

$$\left.\begin{aligned}(1-i)w - iz &= 0\\ 2w + (1-i)z &= 0\end{aligned}\right\}$$

$w = \frac{i}{1-i} z$ (Inserting this into second equation, we get $\underbrace{\left(\frac{2i}{1-i} + 1-i\right)}_{0} z = 0$. So z can take any complex value.

Hence $\operatorname{Nul}(L) = \left\{\left(\frac{i}{1-i} z, z\right) ; z \in \mathbb{C}\right\}$

Why? $= \underbrace{\operatorname{span}\left\{\left(\frac{i}{1-i}, 1\right)\right\}}$

Why? This is a linearly independent set and spans $\operatorname{Nul}(L)$. Hence it is a basis for $\operatorname{Nul}(L)$.

2.b) (4 points) Determine a basis for the range of L.

We will use the fact at the beginning of the page. First note that we can easily extend $\left\{\left(\frac{i}{1-i}, 1\right)\right\}$ to a basis of $\mathbb{C}^2$. For example, to the set $\underbrace{\left\{\left(\frac{i}{1-i}, 1\right), (0,1)\right\}}$

Check that this is a linearly independent set. Since we know $\mathbb{C}^2$ is 2-dimensional (over $\mathbb{C}$), it is a basis of $\mathbb{C}^2$.

Now from the fact at the beginning of the page, we get

$L\left(\left\{\left(\frac{i}{1-i}, 1\right), (0,1)\right\}\right)$ spans $\operatorname{Ran}(L)$.

and this set is equal to $\{(0,0), L((0,1))\}$

since $L\left(\left(\frac{i}{1-i}, 1\right)\right) = (0,0)$ (which we found in the previous part). So

$\operatorname{Ran}(L) = \operatorname{span}\{L((0,1))\}$

$= \operatorname{span}\{(-i, 1-i)\}$

Hence $\{(-i, 1-i)\}$ is a basis for $\operatorname{Ran}(L)$.

2.c) (4 points) Determine if L is one-to-one or onto

A linear operator is one-to-one iff $\operatorname{Nul}(L) = \{0\}$. So L is not one-to-one. Since $\operatorname{Ran}(L)$ is one-dimensional (Why?) but $\mathbb{C}^2$ is two-dimensional, it is also not onto.

Math 107, Fall 2012, Quiz # 4e

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (4 points) Give the definition of the following terms.

1.a) A linear operator: Let V, W be vector spaces over $\mathbb{F}$ and $L: V \to W$ be an operator. L is said to be a linear operator if for all $\alpha \in \mathbb{F}$, $v \in V$, $w \in V$ the following holds: i) $L(v+w) = L(v) + L(w)$
ii) $L(\alpha v) = \alpha L(v)$

1.b) Null space of a linear operator:
Let $L: V$ and W be as in part a. Then null space of L, denoted by $\mathrm{Nul}(L)$ is equal to the set $\{v \in V : L(v) = 0\}$ 0 vector of W

1.c) Inverse image of a subset under a function:
Let $f: A \to B$ be a function and $C \subset B$. Then inverse image of C under f, denoted by $f^{-1}(C)$ is equal to the set $\{a \in A : f(a) \in C\}$

1.d) Image of a subset under a function:
Let $f: A \to B$ be a function and $C \subset A$. Then image of C under f, denoted by $f(C)$ is equal to the set $\{b \in B : \exists c \in C$ with $f(c) = b\}$

Problem 2 (16 points) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be an everywhere-defined function given by $L(x, y) := (2x - 3y, -y)$. Given that $U := \{(x, -x,) | x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$,

2.a) Show that L is a linear operator.

i) Let (x_1, y_1) and (x_2, y_2) be two elements from $\mathbb{R}^2$.

Then $L((x_1,y_1)) + L((x_2,y_2)) = (2x_1 - 3y_1, -y_1) + (2x_2 - 3y_2, -y_2)$
$= (2(x_1+x_2) - 3(y_1+y_2), -(y_1+y_2))$
$= L((x_1+x_2, y_1+y_2))$
$= L((x_1,y_1) + (x_2,y_2))$
$\Rightarrow$ So, condition (i) of part 1.a is satisfied

ii) Let $\alpha \in \mathbb{R}$ and $(x,y) \in \mathbb{R}^2$. Then
$L(\alpha(x,y)) = L((\alpha x, \alpha y)) = (2\alpha x - 3\alpha y, -\alpha y)$
$= \alpha(2x - 3y, -y)$
$= \alpha L((x,y))$
$\Rightarrow$ So condition (ii) of part 1.a. is satisfied and L is linear operator

2.b) Determine a basis for the image of U under L.

$L(U) := \{(2x - 3(-x), -(-x)) : x \in \mathbb{R}\}$

$= \{(5x, x) : x \in \mathbb{R}\}$

$= \text{span}\underbrace{\{(5,1)\}}$

This set is linearly independent. (Why?)

This set spans $L(U)$. (Why?)

Hence $\{(5,1)\}$ is a basis for $L(U)$.

2.c) Determine a basis for the inverse image of U under L.

Note that $(x,y) \in L^{-1}(U)$ if and only if

$L(x,y) = (2x-3y, -y) \in U$ which holds if and only if

$2x - 3y = -(-y) \implies x = 2y$

So

$L^{-1}(U) = \{(2y, y) : y \in \mathbb{R}\}$

$= \text{span}\{(2,1)\}$

The fact that $\{(2,1)\}$ is a basis for $L^{-1}(U)$ follows as in the preceding exercise.

2.c) Find the null space of L. Is the linear operator L one-to-one? Explain why.

Note that $(x,y) \in Nul(L)$ iff $L((x,y)) = 0$

So $(x,y) \in Nul(L)$ iff $2x - 3y = 0$ and $-y = 0$.

This implies that $Nul(L) = \{(0,0)\}$

Hence the linear operator L is one-to-one.

Thm: A linear operator is one-to-one iff $Nul(L) = \{0\}$.

Math 107, Fall 2012, Quiz # 4f

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (3 points) Give the definition of the following terms.

1.a) A linear operator:

Quiz 4e/1.a

1.b) Image of a subset under a function:

Quiz 4e/1.d

1.c) Range of a function:

Quiz 4d/1.c

Problem 2 (8 points) Let $L : \mathbb{C}^2 \to \mathbb{C}^2$ be an everywhere-defined linear operator given by $L(w, z) := ((1-i)w - iz, w + iz)$ ~~and $U := \{(0, iz) | z \in \mathbb{C}\}$~~. Determine the null space and range of L.

$(w, z) \in Nul(L)$ if and only if

$L(w,z) = ((1-i)w - iz, w + iz) = (0,0)$

$$\Rightarrow \left.\begin{array}{r} (1-i)w - iz = 0 \\ w + iz = 0 \end{array}\right\} \; w = 0, \; z = 0$$

$\Rightarrow Nul(L) = \{(0,0)\}$

We know that

$dim(Dom(L)) \geq dim(Nul(L)) + dim(Ran(L))$

Since L is everywhere defined, the equality above gives us

$2 = 0 + dim(Ran(L))$

$\boxed{\text{Since } dim(Ran(L)) = 2, \; Ran(L) = \mathbb{C}^2.}$ Why?

Note: In the following, vector spaces consist of specific functions - polynomials -. In order to show that two elements are equal in these spaces, we have to show the equality of functions. And two functions are equal if $f(x) = g(x)$ for all $x \in$ Dom f = Dom g

Problem 3 (9 points) Let $\mathcal{P}(\mathbb{R},\mathbb{R})$ be the vector space of real polynomials. Determine whether the functions $L : \mathcal{P}(\mathbb{R},\mathbb{R}) \to \mathcal{P}(\mathbb{R},\mathbb{R})$ are linear operators or not. Here $p \in \mathcal{P}(\mathbb{R},\mathbb{R})$ and $x \in \mathbb{R}$ and p' denotes the derivative of p. Justify your response.

Notation: For an operator L, $Lp := L(p)$. These notations can be used interchangeably

3.a) $(Lp)(x) := p'(0) \int_0^x p(t)dt$

Let $p, q \in P(\mathbb{R},\mathbb{R})$ and $\alpha \in \mathbb{R}$.

i) We will first show that $\underbrace{L(p+q) = L(p) + L(q)}$ Equality of two functions!!

Let $x \in \mathbb{R}$. Then

$$(L(p+q))(x) = (p'(0) + q'(0)) \int_0^x (p(t) + q(t))dt$$

$$= p'(0)\int_0^x p(t)dt + q'(0)\int_0^x q(t)dt + p'(0)\int_0^x q(t)dt + q'(0)\int_0^x p(t)dt$$

$A(x)$

$$= (L(p))(x) + (L(q))(x) + A(x)$$

Note that for L to be linear, $A(x)$ has to be 0 for all $x \in \mathbb{R}$. Find $p, q \in P(\mathbb{R},\mathbb{R})$ such that $A(x) \neq 0$ so that L is not linear

3.b) $(Lp)(x) := 2p(1) - p(2)~~dx~~$

Let $p, q \in P(\mathbb{R},\mathbb{R})$ and $\alpha \in \mathbb{R}$.

i) $$(L(p+q))(x) = 2(p(1)+q(1)) - (p(2)+q(2))$$
$$= (2p(1) - p(2)) + (2q(1) - q(2))$$
$$= (L(p))(x) + (L(q))(x)$$

ii) It is easy to check $L(\alpha p) = \alpha L(p)$.

3.c) $(Lp)(x) := \dfrac{p(x)}{x^2+1}$

i) $$(L(p+q))(x) = \frac{p(x)+q(x)}{x^2+1}$$
$$= \frac{p(x)}{x^2+1} + \frac{q(x)}{x^2+1}$$
$$= (L(p))x + (L(q))(x)$$

ii) $$(L(\alpha p))(x) = \frac{\alpha p(x)}{x^2+1} = \alpha(L(p))(x)$$

So L is a linear operator.