Math 107, Fall 2012, Quiz # 6a

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (12 points) Let $D, K : \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ be linear operators defined by

$$\forall f \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}),\ \forall x \in \mathbb{R},\ (Df)(x) := \frac{d}{dx}f(x),\ (Kf)(x) := \int_0^x f(t)dt.$$

Calculate $((KD - DK)f)(x)$ to show that $KD \neq DK$.

Quiz 6e – Problem 1

Problem 2 (8 points) Find an isomorphism $\phi := \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(E_2) = B$, where $B := \{(1,1),(0,-1)\}$.

Let $\phi(1,0) = (1,1)$

$\phi(0,1) = (0,-1)$

Since $\phi(x,y) = x\,(\phi(1,0) + y\,\phi(0,1)$ $\forall x,y \in \mathbb{R}$

$= x(1,1) + y(0,-1)$

$\phi(x,y) = (x, x-y)$

Math 107, Fall 2012, Quiz # 6b

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (8 points) Find an element L of $\mathcal{L}(\mathbb{R}^2)$ such that $L \neq I$ and $L^2 = I$. ($L^2 := L \circ L$)

Let $(x,y) \in \mathbb{R}^2$, since L is a linear operator

$L(x,y) = xL(1,0) + yL(0,1)$

$L^2 = I \Rightarrow L^2(1,0) = (1,0)$ $\Rightarrow$ Choose $L(1,0) = (0,1)$

$L^2(0,1) = (0,1)$ $\qquad L(0,1) = (1,0)$

Then $L(x,y) = (y,x)$ and $L^2(x,y) = (x,y)$

however $L(x,y) \neq (x,y)$ $\forall (x,y) \in \mathbb{R}$

Problem 2 (12 points) Let $L := \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $L(x,y) := (x+y, 2x-y, x-y)$. Show that L is an invertible operator.

— $L(x,y)$ is a linear operator, since

- $\text{Dom}\, L = \mathbb{R}^2$ (everywhere defined)
- $L(\alpha(x,y) + \beta(\bar{x},\bar{y})) = (\alpha x + \beta\bar{x} + \alpha y + \beta\bar{y},\ 2(\alpha x + \beta\bar{x}) - (\alpha y + \beta\bar{y}),$
  $\alpha x + \alpha\bar{x} - (\alpha y + \alpha\bar{y}))$
  $= \alpha(x+y, 2x-y, x-y) + \beta(\bar{x}+\bar{y}, 2\bar{x}-\bar{y}, \bar{x}-\bar{y})$
  for $(x,y), (\bar{x},\bar{y}) \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$.

not necessary if it is given then L is a linear operator

— $L(x,y)$ is an invertible linear operator. Because

$(x+y, 2x-y, x-y) = (0,0,0)$

$\Rightarrow (x,y,z) = (0,0,0)$

which means $\text{Null}\, L = \{(0,0,0)\} = \{\vec{0}_{\mathbb{R}^3}\}$

Math 107, Fall 2012, Quiz # 6c

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (10 points) Find an element L of $\mathcal{L}(\mathbb{R}^2)$ such that $L \neq I$ and $L^2 = I$. ($L^2 := L \circ L$)

Problem 1 – Quiz 6b

Problem 2 (10 points) Let $U := \{[M_{ij}] \in \mathfrak{M}(2,2;\mathbb{F}) | M_{11} + M_{22} = 0, M_{12} = M_{21}\}$. Show that U is a subspace of $\mathfrak{M}(2,2;\mathbb{F})$ and determine its dimension.

(i) • $\mathcal{U}$ is nonempty since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{U}$

• Let M and $\tilde{M}$ be elements of $\mathcal{U}$

Then $(M+\tilde{M})_{11} + (M+\tilde{M})_{22} = \underbrace{M_{11} + M_{22}}_{0} + \underbrace{\tilde{M}_{11} + \tilde{M}_{22}}_{0}$

$= 0$

and $(M+\tilde{M})_{12} = M_{12} + \tilde{M}_{12} = M_{21} + \tilde{M}_{21}$

$= (M+\tilde{M})_{21}$

So $(M+\tilde{M}) \in \mathcal{U}$, i.e. $\mathcal{U}$ is closed under addition

• Let $M \in \mathcal{U}$ and $\alpha \in \mathbb{F}$, then $(\alpha M)_{ij} = \alpha M_{ij}$

$\left.\begin{array}{l} \alpha M_{11} + \alpha M_{22} = 0 \\ \alpha M_{12} = \alpha M_{21} \end{array}\right\} \Rightarrow (\alpha M) \in \mathcal{U}$

i.e. $\mathcal{U}$ is closed under scalar multiplication.

(ii) Dim $\mathcal{U} = ?$ $\mathcal{U} = \left\{ \begin{bmatrix} a & b \\ b & -a \end{bmatrix} : a, b \in \mathbb{F} \right\} = \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \, a, b \in \mathbb{R} \right\}$

Dim $\mathcal{U} = 2$

Math 107, Fall 2012, Quiz # 6d

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (10 points) Let V be a vector space over $\mathbb{F}$ and $L \in \mathcal{L}(V)$ such that $L \neq I$ and $L^2 = I$. Show that $Nul(L) := \{o\}$ and $Ran(L) = V$.

$$Null(L) := \{v \in V : L(v) = 0\} = \{v \in V : L^2(v) = L(0)\}$$
$$= \{v \in V : I(v) = L(0)\}$$
$$= \{0\}, \text{ since } L \text{ is a linear operator } (L(0) = 0)$$

Since $\underbrace{\dim V}_{= (\mathrm{Dom} L)} = \dim Null(L) + \dim Ran(L)$, since $L \in \mathcal{L}(V)$

$\dim V = \dim Ran(L) \Rightarrow V = Ran(L)$

Problem 2 (10 points) Let $U := \{M_{ij} \in \mathfrak{M}(2,2;\mathbb{F}) | M_{11} + M_{22} = 0, M_{12} = M_{21}\}$. Show that U is a subspace of $\mathfrak{M}(2,2;\mathbb{F})$ and determine its dimension.

Problem 2 – Quiz 6c

Math 107, Fall 2012, Quiz # 6e

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (12 points) Let $D, K : \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ be linear operators defined by

$$\forall f \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}),\ \forall x \in \mathbb{R},\ (Df)(x) := \frac{d}{dx}f(x),\ (Kf)(x) := \int_0^x f(t)dt.$$

Calculate $((KD - DK)f)(x)$ to show that $KD \neq DK$.

$$K(Df)(x) = K\left(\frac{df}{dx}\right)(x) = \int_0^x \frac{df}{dt}\,dt = f(x) - f(0)$$

$$D(Kf)(x) = D\left(\int_0^x f(t)dt\right) = \frac{d}{dx}\left(\int_0^x f(t)dt\right) = f(x)$$

So $((KD - DK)f)(x) = -f(0)$ which may not be zero.

Therefore $KD \neq DK$

Problem 2 (8 points) Find an element L of $\mathcal{L}(\mathbb{R}^2)$ such that $L \neq I$ and $L^2 = I$. ($L^2 := L \circ L$).

Problem 1 – Quiz 6

Math 107, Fall 2012, Quiz # 6f

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Let $L, K \in \mathcal{L}(\mathbb{C}^2)$ be defined as $L(w,z) := (w+iz, w-iz)$ and $K(w,z) := (w-iz, w+iz)$ $\forall w, z \in \mathbb{C}$.

Problem 1 (15 points) Find the representations of K, L and KL in the standart basis $E_2 := \{(1,0),(0,1)\}$.

Representation of K in E_2

$$K(1,0) = (1,1) = 1(1,0) + 1(0,1)$$
$$K(0,1) = (-i,i) = -i(1,0) + i(0,1)$$

$$[K]_{E_2} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

Explanation:
Note that
$e_1 := (1,0)$
$e_2 := (0,1) \Rightarrow$
$K(e_1) = e_1 + e_2$
$K(e_2) = -ie_1 + ie_2$
$\Rightarrow K_{11} = 1$
$K_{12} = -i$
$K_{21} = i$
$K_{22} = 1$

Representation of L in E_2

$$L(1,0) = (1,1) = 1(1,0) + 1(0,1)$$
$$L(0,1) = (+i,-i) = i(1,0) - i(0,1)$$

$$\Rightarrow [L]_{E_2} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

Representation of KL in E_2

$$[KL]_{E_2} = [K]_{E_2}[L]_{E_2} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

$$= \begin{pmatrix} 1-i & i-1 \\ 1+i & i+1 \end{pmatrix}$$

Problem 2 (5 points) Show that L is an invertible operator.

$\text{Null}(L) = \{0\}$, since $(w+iz, w-iz) = (0,0)$

$\Rightarrow w = 0, z = 0$

also $L \in \mathcal{L}(\mathbb{C}^2)$, therefore L is an invertible operator