Math 107, Fall 2012, Quiz # 7a

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (5 points) Give an example of a real 2×2 matrix such that $\text{Rank}(\mathbf{M}) = \text{Nullity}(\mathbf{M})$.

$$M = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \left(\begin{array}{l} \text{Rank}(M) = 1 \\ \text{Nullity}(M) = 1 \end{array} \right)$$

Problem 2 (3 points) Find the matrices $\mathbf{A}$ and $\mathbf{b}$ such that the following system of equations takes the form $\mathbf{Ax} = \mathbf{b}$.

$$3x - iy = 0,$$
$$6ix + 2y = -1.$$

$$A = \begin{bmatrix} 3 & -i \\ 6i & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Problem 3 (12 points) For the matrices $\mathbf{A}$ and $\mathbf{b}$ that you find in response to the preceding problem, compute rank of $\mathbf{A}$ and $[\mathbf{A}|\mathbf{b}]$, and adress the problems of existence and uniqueness of the solution of the corresponding system of equations.

- The columns of A are $\begin{bmatrix} 3 \\ 6i \end{bmatrix}$, $\begin{bmatrix} -i \\ 2 \end{bmatrix}$. Since $\begin{bmatrix} 3 \\ 6i \end{bmatrix} = 3i \begin{bmatrix} -i \\ 2 \end{bmatrix}$

Therefore the columns of A are linearly dependent ~~on~~ $\text{Rank}(A) = 1$

- The columns of $[A|b]$ are $\begin{bmatrix} 3 \\ 6i \end{bmatrix}$, $\begin{bmatrix} -i \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Since

$\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is not a scalar multiple of $\begin{bmatrix} -i \\ 2 \end{bmatrix}$, There are 2 linearly independent columns. $\text{Rank}([A|b]) = 2$.

$\text{Rank}([A|b]) \neq \text{Rank}([A]) \Rightarrow$ There exists no soln.

Math 107, Fall 2012, Quiz # 7b

You have 25 minutes.

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☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (10 points) Show that for a real or complex 2×2 matrix **M**, Rank(**M**)=1 iff $\mathbf{M} \neq 0$ and det(**M**)=0.

If Rank(M) = 1, since M consists of 2 columns, the columns are linearly dependent, which means that det M = 0 and M ≠ 0 (If M = 0, then Rank(M) = 0)
If M ≠ 0 then Rank(M) > 0, We also know Rank(M) ≤ 2
If M ≠ 0 and det(M) = 0 then Rank(M) ≠ 2.
So Rank(M) = 1.

Problem 2 (10 points) Find all real numbers α and β such that the following system of equations does not have a solution.

$$x + 3y + \alpha z = \beta,$$
$$2x + \alpha y + 12z = \alpha - 2.$$

Quiz 7e, Problem 1.

Math 107, Fall 2012, Quiz # 7c

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (10 points) Find the rank and nullity of the matrix **A** given below.

$$\mathbf{A} := \begin{bmatrix} 2 & -3 & 1 & 0 \\ -2 & 4 & 0 & 1 \end{bmatrix} \tag{1}$$

Since the columns $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent and Rank $A \leq 2$, Rank$(A) = 2$. By the Rank-Nullity theorem Nullity$(A) = 0$.

Problem 2 (10 points) Use Cramer's Rule to solve the following system of equations:

$$2x + 3iy = -1,$$
$$3x + 4iy = i.$$

Equivalently,

$$\begin{bmatrix} 2 & 3i \\ 3 & 4i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix}$$

$$x = \frac{\det \begin{bmatrix} -1 & 3i \\ i & 4i \end{bmatrix}}{\det \begin{bmatrix} 2 & 3i \\ 3 & 4i \end{bmatrix}} = \frac{-4i + 3}{-i} = 4 + 3i$$

$$y = \frac{\det \begin{bmatrix} 2 & -1 \\ 3 & i \end{bmatrix}}{\det \begin{bmatrix} 2 & 3i \\ 3 & 4i \end{bmatrix}} = \frac{2i + 3}{-i} = -2 + 3i$$

Math 107, Fall 2012, Quiz # 7d

You have 25 minutes.

Name, Last Name: Student No: Signature:

☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (10 points) Show that for a real or complex 2×2 matrix $\mathbf{M}$, Rank($\mathbf{M}$)=0 iff $\mathbf{M} = 0$ and Rank($\mathbf{M}$)=2 iff det($\mathbf{M}$) $\neq 0$.

$Rank(M) \leq 2$

If $M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $Rank(M) = 0$ since the columns are generated by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. If Rank $M = 0$, then the dimension of the vector space generated by those columns of $M = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$. Therefore $M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

If $Rank(M) = 2$ then the columns of M are linearly independent, which means that $\det(M) \neq 0$. If $\det(M) \neq 0$ then the columns are linearly independent, which means $Rank(M) = 2$.

Problem 2 (10 points) Find the rank and nullity of the matrix $\mathbf{A}$ given below.

$$\mathbf{A} := \begin{bmatrix} 1+i & 2i \\ 2-2i & -2i \\ 1-i & 2 \end{bmatrix} \tag{1}$$

$Rank(A) = 2$, because there is no scalar α s.t.

$$\alpha \begin{bmatrix} 2i \\ -2i \\ 2 \end{bmatrix} = \begin{bmatrix} 1+i \\ 2-2i \\ 1-i \end{bmatrix}$$

$Nullity(A) = 0$, since $Rank(A) + Nullity(A) = 2$.

Math 107, Fall 2012, Quiz # 7e

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☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (10 points) Find all real numbers α and β such that the following system of equations does not have a solution.

$$x + 3y + \alpha z = \beta,$$
$$2x + \alpha y + 12z = \alpha - 2.$$

Equivalently,

$A\vec{x} = \vec{b}$, where $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $A = \begin{bmatrix} 1 & 3 & \alpha \\ 2 & \alpha & 12 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} \beta \\ \alpha - 2 \end{bmatrix}$

System has no soln iff $\text{Rank}(A) \neq \text{Rank}([A|b])$

Choose $\alpha = 6$, then all the columns of A are scalar multiple of each other. i.e, $\text{Rank}(A) = 1$. On the other hand $\vec{b} = \begin{bmatrix} \beta \\ 4 \end{bmatrix}$, it is enough to choose β s.t. $\vec{b}$ is not a scalar multiple of the columns of A. Let $\beta = 4$. Then $\text{Rank}[A|b] = 2$, $2 \neq 1$.

Problem 2 (10 points) Use Cramer's Rule to solve the following system of equations:

$$2x + 3y = 1,$$
$$3x + 4y = 2.$$

Equivalently,

$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

$$x = \frac{\det \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}}{\det \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}} = \frac{-2}{-1} = 2$$

$$y = \frac{\det \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}}{\det \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}} = \frac{1}{-1} = -1$$

Math 107, Fall 2012, Quiz # 7f

You have 25 minutes.

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☐ Section 1 (Tue,Thu, & Fr 12:30-13:20) ☐ Section 2 (Tue,Thu, & Fr 14:30-15:20)

Problem 1 (5 points) Give an example of a real 2×2 matrix such that Rank(**M**) = 0.

$$M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 2 (3 points) Find the matrices **A** and **b** such that the following system of equations takes the form $\mathbf{A}\mathbf{x} = \mathbf{b}$.

$$x + 2y = 1,$$
$$y + 2x = 0.$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Problem 3 (12 points) For the matrices **A** and **b** that you find in response to the preceding problem, compute rank of **A** and [**A**|**b**], and adress the problems of existence and uniqueness of the solution of the corresponding system of equations.

- The columns of A are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. They are linearly independent since $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. So $Rank(A) = 2$.
- The columns of [A|b] are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Since $Rank([A|b]) \leq 2$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ linearly independent. $Rank\,[A|b] = 2$.

- Since $Rank(A) = Rank([A|b])$, there exist a solution. and it is a unique soln, because $Rank(A) = 2$ (number of columns of A)