1)
$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & -2 & s & 0 \\ 3 & s & t & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -1 & s-1 & 0 \\ 0 & 3+s & t-3 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1-s & 0 \\ 0 & 3 & -1 & 0 \\ 0 & s+3 & t-3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1-s & 0 \\ 0 & 0 & 3s-4 & 0 \\ 0 & 0 & s^2+2s+t-6 & 0 \end{array}\right)$$

$$(-s-3)(1-s) = -s+s^2-3+3s$$
$$s^2+2s-3+t-3$$
$$s^2+2s+t-6$$

If $3s-4=0 \Leftrightarrow s=\frac{4}{3}$ and $s^2+2s+t-6=0$

$\Leftrightarrow s=\frac{4}{3}$ & $\frac{16}{9}+\frac{8}{3}+t-6=0 \Leftrightarrow t=6-\frac{16}{9}-\frac{8}{3}$

$$=\frac{54-16-24}{9}$$
$$=\frac{14}{9}$$

we have nontrivial solutions.

Problem 2 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

a (10 points) Prove that T is one-to-one if and only if $\mathrm{Nul}(T)$ is trivial, i.e., $\mathrm{Nul}(T) = \{\mathbf{0}\}$.

$\Rightarrow$ If T is 1-to-1, $\forall\, x \in Nul(T)$, $Tx = 0 = T0$

$\Rightarrow x = 0$ because T is 1-to-1 $\Rightarrow Nul(T) \subseteq \{0\}$

But $0 \in Nul(T) \Rightarrow \{0\} \subseteq Nul(T) \Rightarrow Nul(T) = \{0\}$

$\Leftarrow$ If $Nul(T) = \{0\}$, $\forall\, x_1, x_2 \in \mathbb{R}^n$

$T(x_1) = T(x_2) \Rightarrow T(x_1) - T(x_2) = 0 \Rightarrow$

$T(x_1 - x_2) = 0 \Rightarrow x_1 - x_2 \in Nul(T)$

but $Nul(T) = \{0\} \Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2 \Rightarrow T$ is 1-to-1. $\boxtimes$

b (10 points) Show that the range (image) of T is a subspace of $\mathbb{R}^m$.

1) $T(0) = 0 \Rightarrow 0 \in Ran(T) \subseteq \mathbb{R}^m$

2) let $y_1, y_2 \in Ran(T) \Rightarrow \exists\, x_1, x_2 \in \mathbb{R}^n$,

$T(x_1) = y_1$, $T(x_2) = y_2 \Rightarrow y_1 + y_2 = T(x_1) + T(x_2)$

$= T(x_1 + x_2)$

$\Rightarrow y_1 + y_2 \in Ran(T)$

3) let $\alpha \in \mathbb{R}$, $y \in Ran(T) \Rightarrow \exists\, x \in \mathbb{R}^n$, $T(x) = y$

$\Rightarrow \alpha y = \alpha T(x) = T(\alpha x) \Rightarrow \alpha y \in Ran(T)$

1, 2, 3 $\Rightarrow Ran(T)$ is a subspace of $\mathbb{R}^m$.

3)

a) $$3\left\{\underbrace{\begin{pmatrix} \boxed{\text{raw } 2} \\ \\ \\ \end{pmatrix}}_{n}\right. \longrightarrow \begin{bmatrix} \boxed{2 \cdot \text{raw } 1 + \text{raw } 2} \\ \\ \\ \end{bmatrix}$$

$n = 1$

$$E: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ 2x_1 + x_2 \\ x_3 \end{pmatrix}$$

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Notes

E is linear since $E(x+y) = E\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix}$

$= \begin{pmatrix} x_1+y_1 \\ 2x_1+2y_1+x_2+y_2 \\ x_3+y_3 \end{pmatrix} = E(x)+E(y)$. $E(\alpha x) = E\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix}$

$= \begin{pmatrix} \alpha x_1 \\ 2\alpha x_1 + \alpha x_2 \\ \alpha x_3 \end{pmatrix} = \alpha \cdot E(x)$

---

b) $E(x) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot X$

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c) $E^{-1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -2x_1+x_2 \\ x_3 \end{pmatrix}$

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4) $A = \begin{pmatrix} -1 & 2 & a \\ 2 & -4 & -1 \\ 3 & 0 & 2a \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & a \\ 0 & 0 & 2a-1 \\ 0 & 6 & 8a \end{pmatrix}$

$\sim \begin{pmatrix} -1 & 2 & a \end{pmatrix}$

$$\sim \begin{pmatrix} -1 & 2 & a \\ 0 & 6 & 8a \\ 0 & 0 & 2a-1 \end{pmatrix}$$

If $2a-1 \neq 0$, A is invertible since it is reducible to $Id^{3\times 3}$.

So $a \neq \frac{1}{2}$

b) $a = 0$

$$\left(\begin{array}{ccc|ccc} -1 & 2 & 0 & 1 & 0 & 0 \\ 2 & -4 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 0 & 0 \\ 2 & -4 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 & 0 \\ 0 & 6 & 0 & 3 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 3/6 & 0 & 1/6 \\ 0 & 0 & -1 & 2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 2/6 \\ 0 & 1 & 0 & 3/6 & 0 & 1/6 \\ 0 & 0 & 1 & -2 & -1 & 0 \end{array}\right)$$

$$A^{-1} = \frac{1}{6}\begin{pmatrix} 0 & 0 & 2 \\ 3 & 0 & 1 \\ -12 & -6 & 0 \end{pmatrix}$$

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5) $$T\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 2x+y \\ z \\ x+z+t \end{pmatrix} \qquad T: \mathbb{R}^4 \to \mathbb{R}^3$$

$$T(x) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \cdot x$$

$$\left(\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1/2 & 1 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -2 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

$x = -t$

$y = 2t$

$z = 0$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

So $\text{Null}(T) = \text{Span}\left\{ \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$

b) $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$

c) $A^T = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad S(x) = A^T \cdot x$

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

All columns are lin. ind. So a basis of $Col(A^T)$

$$= \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

---

d) S is 1-1 since A^T is reduced to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which has a pivot in each column. So no free variables in the solution of $A^T x = 0$.
So the only solution is $x = 0$