

Problem 1 (15 points) Find the explicit solution of the following initial-value problem.

$$2xyy' + y^2 + 2x = 0, \quad y(1) = 1.$$

$$\text{Let } u := y^2 \quad \hookrightarrow \quad xu' + u + 2x = 0$$

$$\Rightarrow \quad u' + \frac{u}{x} = -2$$

$$\Rightarrow \quad \mu u' + \frac{\mu}{x} u = -2\mu$$

$$\frac{d}{dx}(\mu u) + \underbrace{\left[-\mu' + \frac{\mu}{x}\right]}_0 u = -2\mu$$

$$\frac{\mu'}{\mu} = \frac{1}{x} \Rightarrow \mu = ax \quad \text{take } a=1 \Rightarrow \underline{\mu(x) = x}$$

$$\Rightarrow \quad \frac{d}{dx}(xu) = -2x$$

$$\Rightarrow \quad xu = -x^2 + c$$

$$\Rightarrow \quad u = -x + \frac{c}{x}$$

$$\Rightarrow \quad y^2 = -x + \frac{c}{x}$$

$$\Rightarrow \quad y = \pm \sqrt{-x + \frac{c}{x}}$$

$$y(1) = 1 \quad \hookrightarrow \quad 1 = \pm \sqrt{-1 + c}$$

$$\Rightarrow \quad \text{choose } + \text{ \& } c = 2$$

$$\Rightarrow \quad \boxed{y = \sqrt{-x + \frac{2}{x}}}$$

Problem 2 (10 points) Let $y(t)$ be a twice differentiable function with Laplace transform $Y(s)$ for $s > 0$. Show that for $s > 0$ the Laplace transform of $t^2 y''(t)$ is given by

$$\mathcal{L}\{t^2 y''(t)\} = s^2 Y''(s) + 4s Y'(s) + 2Y(s).$$

$$\mathcal{L}\{t^2 y''(t)\} = \int_0^{\infty} e^{-st} t^2 y''(t) dt$$

$$= \int_0^{\infty} \left(\frac{\partial^2}{\partial s^2} e^{-st} \right) y''(t) dt$$

$$= \frac{d^2}{ds^2} \mathcal{L}\{y''(t)\}$$

$$= \frac{d^2}{ds^2} [s^2 Y(s) - y'(0) - sy(0)]$$

$$= \frac{d}{ds} [2s Y(s) + s^2 Y'(s)]$$

$$= 2 Y(s) + 2s Y'(s) + 2s Y'(s) + s^2 Y''(s)$$

$$= s^2 Y''(s) + 4s Y'(s) + 2 Y(s).$$

Problem 3 (20 points) Consider the equation $y''(t) + 2ay'(t) + by(t) = 0$ where a and b are real numbers.

3.a) (5 points) Reduce this equation to a system of first order equations.

$$\begin{aligned}
 x_1 &:= y \\
 x_2 &:= y' \quad \Rightarrow \quad x_2' = y'' = -(2ay' + by) = -2ax_2 - bx_1
 \end{aligned}$$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = -bx_1 - 2ax_2 \end{cases} \quad \vec{x} := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \vec{x}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -b & -2a \end{bmatrix}}_A \vec{x}$$

3.b) (15 points) Find conditions on a and b such that all solutions of the system of equations you find in part a of this problem tend to zero as $t \rightarrow +\infty$.

$$0 = \det(A - rI) = \det \begin{bmatrix} -r & 1 \\ -b & -2a-r \end{bmatrix} = r(r+2a) + b$$

$$r^2 + 2ar + b = 0 \Rightarrow r = -a \pm \sqrt{a^2 - b}$$

① If $a^2 = b \Rightarrow \vec{x}(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} e^{-at} \rightarrow 0$ for $t \rightarrow +\infty$ if and only if $\boxed{a > 0}$

② If $a^2 \neq b \Rightarrow \vec{x}(t) = c_1 e^{(-a + \sqrt{a^2 - b})t} + c_2 e^{(a + \sqrt{a^2 - b})t}$

(2.1) If $a^2 > b$, $\vec{x}(t) \rightarrow 0$ as $t \rightarrow +\infty$ provided that

$$-a + \sqrt{a^2 - b} < 0 \Rightarrow -a + \sqrt{a^2 - b} < 0 \Rightarrow \sqrt{a^2 - b} < a \Rightarrow$$

$$\Rightarrow a > 0 \text{ and } a^2 - b < a^2 \Leftrightarrow b > 0$$

So in this case $\boxed{a > 0 \text{ and } b > 0}$

(2.2) If $a^2 < b$, then $\vec{x}(t) \rightarrow 0$ for $t \rightarrow +\infty$ provided that

$$-a < 0 \Rightarrow \boxed{a > 0}$$

$\boxed{\text{So } a > 0 \text{ and } b > 0 \text{ if } a^2 > b.}$

Problem 4 (10 points) Calculate the coefficients of the Fourier series for the function $f: [-\pi, \pi] \rightarrow \mathbb{R}$ given by $f(x) = \pi - |x|$.

f is even \Rightarrow coefficients of sine series are zero.

So the Fourier series for $f(x)$ is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$

where $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} [\pi - |x|] \cos(nx) dx$.

$$\Rightarrow a_n = \underbrace{\int_{-\pi}^{\pi} \cos(nx) dx}_{I_0} + \underbrace{\frac{1}{\pi} \int_{-\pi}^0 x \cos(nx) dx}_{I_1} + \underbrace{\frac{1}{\pi} \int_0^{\pi} -x \cos(nx) dx}_{I_2}$$

For $n=0$: $I_0 = 2\pi$

For $n \geq 1$: $I_0 = \frac{\sin(nx)}{n} \Big|_{-\pi}^{\pi} = 0$

$$\left\{ \begin{aligned} \int x \cos(\alpha x) dx &= \frac{2}{2\alpha} \int \sin(\alpha x) dx = \frac{2}{2\alpha} \left[-\frac{\cos(\alpha x)}{\alpha} \right] + c \\ &= \frac{x \sin(\alpha x)}{\alpha} + \frac{\cos(\alpha x)}{\alpha^2} + c \quad \text{for } \alpha \neq 0 \\ \int x dx &= \frac{x^2}{2} + c \end{aligned} \right.$$

For $n=0$: $I_1 = \frac{x^2}{2} \Big|_{-\pi}^0 = -\frac{\pi^2}{2}$, $I_2 = -\frac{x^2}{2} \Big|_0^{\pi} = -\frac{\pi^2}{2}$

$\Rightarrow a_0 = 2\pi + \frac{1}{\pi} (-\pi^2) = \pi \quad \Rightarrow \boxed{a_0 = \pi}$

For $n > 0$: $I_1 = \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \Big|_{-\pi}^0 = \frac{1}{n^2} - \left(\frac{\cos(-n\pi)}{n^2} \right) = \frac{1 - (-1)^n}{n^2}$

& $I_2 = -\left(\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right) \Big|_0^{\pi} = -\left(\frac{\cos(n\pi)}{n^2} - \left(\frac{1}{n^2} \right) \right) = \frac{1 - (-1)^n}{n^2}$

$\Rightarrow a_n = \frac{1}{\pi} \left[\frac{2(1 - (-1)^n)}{n^2} \right] \Rightarrow \boxed{a_n = \frac{2[1 - (-1)^n]}{\pi n^2} \text{ for } n \geq 1}$

Problem 5 Consider the boundary-value (eigenvalue) problem

$$y''(x) = \lambda y(x), \quad x \in (0, 2\pi)$$

$$y'(0) = 0, \quad y'(2\pi) = 0.$$

where λ is a real number.

5.a (15 points) Find all values of λ for which this problem has a nonzero solution and determine these solutions, i.e., find the eigenvalues and eigenfunctions.

Warning: You must give a detailed justification of your response. Giving the values of λ without explaining how you obtain them and why there are no other possible values will not earn you any credit.

There are 3 possibilities:

(1) $\lambda > 0 \Rightarrow$ let $\mu := \sqrt{\lambda} > 0$ & $y'' - \mu^2 y = 0 \Rightarrow$
 $r^2 - \mu^2 = 0 \Rightarrow r = \pm \mu \Rightarrow y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$
 $\Rightarrow y'(x) = \mu (c_1 e^{\mu x} - c_2 e^{-\mu x})$

$y'(0) = 0 \Rightarrow \mu (c_1 - c_2) = 0 \stackrel{\mu \neq 0}{\Rightarrow} c_1 = c_2$

$y'(2\pi) = 0 \Rightarrow \mu (c_1 e^{2\pi\mu} - c_2 e^{-2\pi\mu}) = 0 \Rightarrow c_1 (e^{2\pi\mu} - e^{-2\pi\mu}) = 0$

$c_1 \neq 0$ because $c_1 = 0 \Rightarrow c_2 = 0 \Rightarrow y = 0$ which is not allowed

$\Rightarrow e^{2\pi\mu} = e^{-2\pi\mu} = 1 \Rightarrow e^{4\pi\mu} = 1 \Rightarrow \mu = 0$ but $\mu > 0$

No solution for $\lambda > 0$.

(2) $\lambda = 0 \Rightarrow y'' = 0 \Rightarrow y = ax + b \Rightarrow y' = a$

$y'(0) = 0 \Rightarrow a = 0 \Rightarrow y(2\pi) = 0 \checkmark \quad \boxed{y(x) = b}$

So $\lambda = 0$ is an eigenvalue

with eigenfunction $y(x) = b$ when $b \in \mathbb{R} \setminus \{0\}$.

(3) $\lambda < 0 \Rightarrow$ let $\mu := \sqrt{-\lambda} > 0$ & $y'' + \mu^2 y = 0 \Rightarrow$

$r^2 = -\mu^2 \Rightarrow r = \pm i\mu \Rightarrow y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$

$\Rightarrow y'(x) = \mu [-c_1 \sin(\mu x) + c_2 \cos(\mu x)]$

$y'(0) = 0 \Rightarrow \mu c_2 = 0 \stackrel{\mu \neq 0}{\Rightarrow} c_2 = 0 \Rightarrow y' = -\mu c_1 \sin(\mu x)$

$y'(2\pi) = 0 \Rightarrow -\mu c_1 \sin(2\pi\mu) = 0 \stackrel{\mu \neq 0 \& c_1 \neq 0}{\Rightarrow} 2\pi\mu = \pi n \text{ for } n \in \mathbb{Z}$

\Downarrow
 $\mu = \frac{n}{2} \Rightarrow \lambda = -\frac{n^2}{4}$

and $\boxed{y(x) = c_1 \cos\left(\frac{n x}{2}\right)} \quad c_1 \in \mathbb{R} \setminus \{0\}$.

5.b (5 points) Show that the eigenfunctions with different eigenvalues are linearly independent.

let $\lambda_1 = -\frac{n_1^2}{4}$, $\lambda_2 = -\frac{n_2^2}{4}$, $n_1, n_2 \in \mathbb{N} = \{0, 1, 2, \dots\}$

be any pair of eigenvalues with eigenfunctions

$$y_1 = a_1 \cos\left(\frac{n_1 x}{2}\right) \text{ and } y_2 = a_2 \cos\left(\frac{n_2 x}{2}\right), a_1, a_2 \neq 0$$

let $\lambda_1 \neq \lambda_2 \Rightarrow n_1 \neq n_2$.

if $\exists b_1, b_2 \in \mathbb{R}$ such that

$$b_1 y_1 + b_2 y_2 = 0 \quad (1)$$

$$\Rightarrow b_1 y_1(x) + b_2 y_2(x) = 0, \quad \forall x \in (0, 2\pi)$$

Recall that $\int_0^{2\pi} \cos\left(\frac{n_1 x}{2}\right) \cos\left(\frac{n_2 x}{2}\right) dx = 0$ because $n_1 \neq n_2$

$$\Rightarrow \int_0^{2\pi} y_1(x) y_2(x) dx = 0 \quad (2)$$

$$\Rightarrow 0 = \int_0^{2\pi} [b_1 y_1(x) + b_2 y_2(x)] y_1(x) dx =$$

$$= b_1 \int_0^{2\pi} y_1(x)^2 dx + b_2 \int_0^{2\pi} y_1(x) y_2(x) dx$$

$\neq 0$ because $y_1(x)^2 \geq 0$ & $y_1 \neq 0$.

$$\Downarrow$$

$$b_1 = 0 \stackrel{(1)}{\Rightarrow} b_2 y_2 = 0 \Rightarrow b_2 = 0$$

$y_2 \neq 0$

so $(1) \Rightarrow b_1 = b_2 = 0 \Rightarrow \{y_1, y_2\}$ is linearly-independent

Problem 6 (25 points) Solve the following problem.

$$u_{tt} + u_t = u_{xx}, \quad x \in (0, \pi), \quad t > 0$$

$$u(0, t) = 0, \quad u(\pi, t) = 0, \quad t > 0$$

$$u(x, 0) = 0, \quad u_t(x, 0) = \sin(2x) + 3 \sin(4x), \quad x \in (0, \pi).$$

$$u(x, t) = X(x)T(t) \Rightarrow X T'' + X T' = X'' T$$

$$\frac{T'' + T'}{T} = \frac{X''}{X} = \lambda \Rightarrow \begin{cases} X'' = \lambda X & \textcircled{1} \\ T'' + T' - \lambda T = 0 & \textcircled{2} \end{cases}$$

$$u(0, t) = 0 \Rightarrow X(0) = 0 \quad \textcircled{3}$$

$$u(\pi, t) = 0 \Rightarrow X(\pi) = 0 \quad \textcircled{4}$$

$$\textcircled{1}, \textcircled{3}, \textcircled{4} \Rightarrow \lambda = -n^2, \quad n \in \mathbb{Z}^+, \quad X = a \sin(nx)$$

$$\textcircled{2} \Rightarrow T'' + T' + n^2 T = 0$$

$$r^2 + r + n^2 = 0 \Rightarrow r = \frac{-1 \pm \sqrt{1-4n^2}}{2}$$

$$\text{Because } n \geq 1, \quad 1-4n^2 < 0 \Rightarrow r = -\frac{1}{2} \pm \frac{i}{2} \sqrt{4n^2-1} = -\frac{1}{2} \pm i \sqrt{n^2 - \frac{1}{4}}$$

$$\Downarrow \\ T(t) = e^{-\frac{t}{2}} [c_1 \sin(\sqrt{n^2 - \frac{1}{4}} t) + c_2 \cos(\sqrt{n^2 - \frac{1}{4}} t)]$$

$$\Downarrow \\ u(x, t) = \sum_{n=1}^{\infty} e^{-\frac{t}{2}} [a_n \cos(\sqrt{n^2 - \frac{1}{4}} t) + b_n \sin(\sqrt{n^2 - \frac{1}{4}} t)] \sin(nx)$$

$$u(x, 0) = 0 \Rightarrow \sum_{n=1}^{\infty} a_n \sin(nx) = 0 \Rightarrow \boxed{a_n = 0}$$

$$u_t(x, t) = \sum_{n=1}^{\infty} \left\{ \left(-\frac{1}{2}\right) [a_n \cos(\sqrt{n^2 - \frac{1}{4}} t) + b_n \sin(\sqrt{n^2 - \frac{1}{4}} t)] + \sqrt{n^2 - \frac{1}{4}} [-a_n \sin(\sqrt{n^2 - \frac{1}{4}} t) + b_n \cos(\sqrt{n^2 - \frac{1}{4}} t)] \right\} e^{-\frac{t}{2}} \sin(nx)$$

$$\Downarrow \\ u_t(x, 0) = \sum_{n=1}^{\infty} \sqrt{n^2 - \frac{1}{4}} b_n \sin(nx) \Rightarrow \sum_{n=1}^{\infty} \sqrt{n^2 - \frac{1}{4}} b_n \sin(nx) = \sin(2x) + 3 \sin(4x)$$

$$\Rightarrow \boxed{b_n = 0 \text{ for } n \neq 2 \text{ and } n \neq 4} \\ b_2 = \frac{1}{\sqrt{4 - \frac{1}{4}}} = \frac{2}{\sqrt{15}}, \quad b_4 = \frac{3}{\sqrt{16 - \frac{1}{4}}} = \frac{6}{\sqrt{63}}$$

$$\Rightarrow u(x, t) = e^{-\frac{t}{2}} \left[\frac{2}{\sqrt{15}} \sin\left(\frac{\sqrt{15}t}{2}\right) \sin(2x) + \frac{6}{\sqrt{63}} \sin\left(\frac{\sqrt{63}t}{2}\right) \sin(4x) \right]$$