Problem 1 (20 points) Use the method of variation of parameters to solve the following initial-value problem.

$$y''(t) - y(t) = 1 + e^{2t}, \qquad y(0) = y'(0) = 0.$$

$y'' - y = 0 \Rightarrow \quad r^2 - 1 = 0 \quad \Rightarrow \quad r = \pm 1$

$y_1 = e^t, \quad y_2 = e^{-t}$

$y = e^t u_1 + e^{-t} u_2$

$y' = e^t u_1' + e^{-t} u_2' + e^t u_1 - e^{-t} u_2$

Set $\boxed{e^t u_1' + e^{-t} u_2' = 0}$

$\Rightarrow \quad y'' = e^t u_1' - e^{-t} u_2' + e^t u_1 + e^{-t} u_2$

$\Rightarrow \quad e^t u_1' - e^{-t} u_2' + e^t u_1 + e^{-t} u_2 - (e^t u_1 + e^{-t} u_2) = 1 + e^{2t}$

$\Rightarrow \quad \boxed{e^t u_1' - e^{-t} u_2' = 1 + e^{2t}}$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-t} \\ 1+e^{2t} & -e^{-t} \end{vmatrix}}{\begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix}} = \frac{-e^{-t}(1+e^{2t})}{-2} = \frac{1}{2}(e^{-t} + e^t)$$

$\Rightarrow \quad \boxed{u_1 = \frac{1}{2}(-e^{-t} + e^t) + c_1}$

$$u_2' = \frac{\begin{vmatrix} e^t & 0 \\ e^t & 1+e^{2t} \end{vmatrix}}{-2} = \frac{e^t(1+e^{2t})}{-2} = -\frac{1}{2}(e^t + e^{3t})$$

$\Rightarrow \quad \boxed{u_2 = -\frac{1}{2}\left(e^t + \frac{e^{3t}}{3}\right) + c_2}$

$\Rightarrow \quad y = e^t\left[\frac{1}{2}(-e^{-t} + e^t) + c_1\right] + e^{-t}\left[-\frac{1}{2}\left(e^t + \frac{e^{3t}}{3}\right) + c_2\right]$

$= c_1 e^t + c_2 e^{-t} + \frac{1}{2}\left(-1 + e^{2t} - 1 - \frac{e^{2t}}{3}\right)$

$\boxed{y = c_1 e^t + c_2 e^{-t} - 1 + \frac{2}{3} e^{2t}}$ $\qquad y' = c_1 e^t - c_2 e^{-t} + \frac{2e^{2t}}{3}$

$\Downarrow$

$$y(0)=0 \Rightarrow c_1 + c_2 - 1 + \frac{1}{3} = 0$$

$$c_1 + c_2 = \frac{2}{3}$$

$$y'(0)=0 \Rightarrow c_1 - c_2 + \frac{2}{3} = 0$$

$$\Rightarrow c_1 - c_2 = -\frac{2}{3}$$

$$\Rightarrow \boxed{c_1 = 0, \quad c_2 = \frac{2}{3}}$$

$$\Rightarrow \boxed{y(t) = \frac{2}{3} e^{-t} - 1 + \frac{e^{2t}}{3}}$$

Problem 2 (20 points) Use the method of power series to solve the following initial-value problem.

$$y'' + xy' - y = -1, \qquad y(0) = y'(0) = 1.$$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = -1$$

$m = n-2$

$$\sum_{m=-2}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = -1$$

$n \to m \qquad n \to m$

$m = 0, 1 \Rightarrow 0$

$$\Rightarrow \sum_{m=0}^{\infty} \left[(m+1)(m+2) a_{m+2} + (m-1) a_m\right] x^m = -1$$

$$\Rightarrow 2a_2 - a_0 + \sum_{m=1}^{\infty} \left[(m+1)(m+2) a_{m+2} + (m-1) a_m\right] x^m = -1$$

$$\Rightarrow \boxed{2a_2 - a_0 = -1}$$

$$\boxed{a_{m+2} = -\frac{m-1}{(m+1)(m+2)} a_m \quad \text{for} \quad m \geq 1}$$

$a_0 = y(0) = 1 \qquad a_1 = y'(0) = 1$

$\Rightarrow \boxed{a_2 = 0}$

For $m = 1 \Rightarrow a_3 = 0 \Rightarrow a_5 = a_7 = \cdots = a_{2n+1} = 0 \quad \forall n \geq 0$

For $m = 2$: $a_4 = -\frac{1}{3 \times 4} a_2 = 0 \Rightarrow a_2 = a_4 = \cdots = a_{2n} = 0$

$$\Rightarrow \boxed{y = 1 + x}$$

Problem 3 (10 points) Let a be a real number and $f(t)$ be a function with Laplace transform $F(s)$ for $s > 0$. Suppose that for every positive real number λ, f satisfies: $f(\lambda t) = \lambda^a f(t)$. Show that

$$F(\tfrac{s}{\lambda}) = \lambda^{a+1} F(s).$$

$$F(\tfrac{s}{\lambda}) = \int_0^\infty e^{-\frac{s}{\lambda}t} f(t)\, dt \qquad \tau := \frac{t}{\lambda} \Rightarrow t = \lambda\tau \Rightarrow dt = \lambda\, d\tau$$

$$= \int_0^\infty e^{-s\tau} f(\lambda\tau)\, \lambda\, d\tau$$

$$= \lambda \int_0^\infty e^{-s\tau} \lambda^a f(\tau)\, d\tau$$

$$= \lambda^{a+1} \int_0^\infty e^{-s\tau} f(\tau)\, d\tau$$

$$= \lambda^{a+1} F(s). \qquad \blacksquare$$

Problem 4 (20 points) Find a function $f(t)$ that for all $t > 0$ satisfies:

$$\int_0^t f(\tau)\, e^{t-\tau} d\tau - f(t) - e^{-t} = 0.$$

Hint: Evaluate the Laplace transform of both sides of this equation and use the formula for the convolution integral.

Let $h(t) := \int_0^t f(\tau)\, e^{t-\tau}\, d\tau$, $F(s) := \mathcal{L}^{-1}\{f$

$$\mathcal{L}\{e^t\} = \frac{1}{s-1} \quad \& \quad \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

$$\mathcal{L}\{h(t)\} = \frac{F(s)}{s-1} \quad \Rightarrow$$

$$\frac{F(s)}{s-1} - F(s) - \frac{1}{s+1} = 0$$

$$F(s)\underbrace{\left(\frac{1}{s-1} - 1\right)}_{\frac{1-s+1}{s-1} = \frac{2-s}{s-1}} = \frac{1}{s+1}$$

$$\Rightarrow \quad F(s) = \frac{s-s}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} = \frac{(A+B)s - 2A + B}{(s+1)(s-2)}$$

$$\Rightarrow \quad \begin{array}{l} A + B = -1 \\ -2A + B = 1 \end{array} \Rightarrow \quad 3A = -2 \Rightarrow \quad \begin{array}{l} A = -\frac{2}{3} \\ B = -\frac{1}{3} \end{array}$$

$$\Rightarrow \quad F(s) = -\frac{1}{3}\left[\frac{2}{s+1} + \frac{1}{s-2}\right]$$

$$\Downarrow$$

$$\boxed{f(t) = -\frac{1}{3}\left(2e^{-t} + e^{2t}\right)}$$

Problem 5 (20 points) Give the general solution of the system of equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} := \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix}.$$

$$\vec{x} = e^{rt}\vec{\xi} \Rightarrow \begin{bmatrix} -1-r & 3 \\ -3 & 5-r \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det\begin{bmatrix} -1-r & 3 \\ -3 & 5-r \end{bmatrix} = 0 \Rightarrow (r+1)(r-5)+9=0$$

$$\Rightarrow r^2-4r+4=0 \Rightarrow (r-2)^2=0$$

$$\Rightarrow \boxed{r=2}$$

repeated eigenvalue

$$\Rightarrow \text{Try}: \boxed{\vec{x}(t) = (\vec{a}+t\vec{b})e^{2t}}, \quad \vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\Rightarrow \vec{x}' = (\vec{b}+2\vec{a}+2\vec{b}t)e^{2t} = \begin{bmatrix} 2a_1+b_1+2b_1t \\ 2a_2+b_2+2b_2t \end{bmatrix}e^{2t}$$

$$\mathbb{A}\vec{x} = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} a_1+tb_1 \\ a_2+tb_2 \end{bmatrix}e^{2t} = \begin{bmatrix} -a_1+3a_2+(-b_1+3b_2)t \\ -3a_1+5a_2+(-3b_1+5b_2)t \end{bmatrix}e^{2t}$$

$$\underline{\vec{x}' = \mathbb{A}\vec{x}} \Rightarrow \begin{cases} 2a_1+b_1+2b_1t = -a_1+3a_2+(-b_1+3b_2)t & (1) \\ 2a_2+b_2+2b_2t = -3a_1+5a_2+(-3b_1+5b_2)t & (2) \end{cases}$$

$$(1) \Rightarrow \begin{cases} 2a_1+b_1 = -a_1+3a_2 & \Rightarrow a_2 = a_1+\frac{b_1}{3} \\ 2b_1 = -b_1+3b_2 & \Rightarrow b_2 = b_1 \end{cases}$$

$$(2) \Rightarrow \begin{cases} 2a_2+b_2 = -3a_1+5a_2 & \Rightarrow a_2 = a_1+\frac{b_2}{3} = a_1+\frac{b_1}{3} \checkmark \\ 2b_2 = -3b_1+5b_2 & \Rightarrow b_2 = b_1 \checkmark \end{cases}$$

$$\Rightarrow \vec{x} = \left(\begin{bmatrix} a_1 \\ a_1+\frac{b_1}{3} \end{bmatrix} + t\begin{bmatrix} b_1 \\ b_1 \end{bmatrix}\right)e^{2t}$$

$$\boxed{\vec{x} = a_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + b_1\begin{bmatrix} t \\ \frac{1}{3}+t \end{bmatrix}e^{2t}}$$

$a_1, b_1 \in \mathbb{R}$ are arbitrary real numbers.

Problem 6a (15 points) Find a fundamental matrix $\Psi(t)$ for the system of equations:

$$x' = x - y,$$
$$y' = -x + y.$$

$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $\vec{x}' = A\vec{x}$

$\vec{x} = e^{rt}\vec{\xi}$, $(A - rI)\vec{\xi} = \vec{0}$

$\Rightarrow \begin{bmatrix} 1-r & -1 \\ -1 & 1-r \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \det\begin{bmatrix} 1-r & -1 \\ -1 & 1-r \end{bmatrix} = 0$

$\Rightarrow (r-1)^2 - 1 = 0 \Rightarrow r - 1 = \pm 1 \Rightarrow r = 1 \pm 1 = \begin{cases} 0 =: r_1 \\ 2 =: r_2 \end{cases}$

$\underline{r = r_1 = 0} \Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \xi_1 = \xi_2$

$\Rightarrow \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{\vec{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$\underline{r = r_2 = 2} \Rightarrow \begin{bmatrix} 1-2 & -1 \\ -1 & 1-2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \Big[$

$\Rightarrow \xi_1 + \xi_2 = 0 \Rightarrow \xi_1 = -\xi_2 \Rightarrow \vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \boxed{\vec{x}^{(2)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2}}$

$$\boxed{\Psi(t) = \begin{bmatrix} 1 & e^{2t} \\ 1 & -e^{2t} \end{bmatrix}}$$

Problem 6b (15 points) Use $\Psi(t)$ that you find in part **a** of this problem to solve the initial-value problem:

$$x' = x - y + 1,$$
$$y' = -x + y,$$
$$x(0) = y(0) = 0. \qquad (1)$$

$\vec{g} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow \quad \vec{x}' = A\vec{x} + \vec{g} \qquad \vec{x} = \Psi \vec{u} \Rightarrow$

$\vec{x}' = \Psi' \vec{u} + \Psi \vec{u}' = A\Psi\vec{u} + \Psi\vec{u}'$

$\Rightarrow \Psi \vec{u}' = \vec{g} \Rightarrow \vec{u}' = \Psi^{-1}\vec{g} \Rightarrow \vec{u} = \int \Psi^{-1} g \, dt$

$\Psi^{-1} = \dfrac{1}{-e^{2t} - e^{2t}} \begin{bmatrix} -e^{2t} & -e^{2t} \\ -1 & 1 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 1 & 1 \\ e^{-2t} & -e^{-2t} \end{bmatrix}$

$\Psi^{-1} g = \dfrac{1}{2}\begin{bmatrix} 1 & 1 \\ e^{-2t} & -e^{-2t} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 1 \\ e^{-2t} \end{bmatrix}$

$\Rightarrow \vec{u} = \dfrac{1}{2}\int \begin{bmatrix} 1 \\ e^{-2t} \end{bmatrix} dt = \dfrac{1}{2}\begin{bmatrix} t + c_1 \\ \dfrac{e^{-2t}}{-2} + c_2 \end{bmatrix}$

$\Rightarrow \vec{x} = \begin{bmatrix} 1 & e^{2t} \\ 1 & -e^{2t} \end{bmatrix} \dfrac{1}{2}\begin{bmatrix} t + c_1 \\ -\dfrac{e^{-2t}}{2} + c_2 \end{bmatrix}$

$= \dfrac{1}{2}\begin{bmatrix} t + c_1 - \frac{1}{2} + c_2 e^{2t} \\ t + c_1 + \frac{1}{2} - c_2 e^{2t} \end{bmatrix}$

$\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \quad \dfrac{1}{2}\begin{bmatrix} c_1 + c_2 - \frac{1}{2} \\ c_1 - c_2 + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{matrix} c_1 + c_2 = \frac{1}{2} \\ c_1 - c_2 = -\frac{1}{2} \end{matrix} \Rightarrow \quad \boxed{c_1 = 0, \quad c_2 = \frac{1}{2}}$

$\Rightarrow \vec{x} = \dfrac{1}{2}\begin{bmatrix} t - \frac{1}{2} + \frac{1}{2}e^{2t} \\ t + \frac{1}{2} - \frac{1}{2}e^{2t} \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \boxed{\begin{aligned} x &= \tfrac{t}{2} + \tfrac{1}{4}(e^{2t} - 1) \\ y &= \tfrac{t}{2} - \tfrac{1}{4}(e^{2t} - 1) \end{aligned}}$