Solution to Make-up Exam

Problem 1 (10 points) Solve the following initial-value problem.

$$y'(t) = \frac{x(y^2-4)}{2}, \qquad y(0) = 3.$$

$$\frac{y'}{y^2-4} = \frac{x}{2} \Rightarrow \int \frac{dy}{y^2-4} = \int \frac{x}{2}\,dx$$

$$\frac{1}{y^2-4} = \frac{1}{4}\left(\frac{1}{y-2} - \frac{1}{y+2}\right)$$

$$\Rightarrow \frac{1}{4}\int \left(\frac{1}{y-2} - \frac{1}{y+2}\right) dy = \frac{x^2}{4} + c$$

$$\Rightarrow \frac{1}{4}\left(\ln|y-2| - \ln|y+2|\right) = \frac{x^2}{4} + c$$

$$\ln\left|\frac{y-2}{y+2}\right| = x^2 + 4c$$

$$\Rightarrow \boxed{\frac{y-2}{y+2} = \underbrace{\pm e^{4c}}_{k} e^{x^2} = k e^{x^2}}$$

$$\Rightarrow \frac{y-2}{4} = \frac{k e^{x^2}}{1-k e^{x^2}}$$

$$\Rightarrow y = 2 + \frac{4k e^{x^2}}{1-k e^{x^2}} = \frac{2+2k e^{x^2}}{1-k e^{x^2}}$$

$$\Rightarrow \boxed{y = \frac{2(1+k e^{x^2})}{1-k e^{x^2}}}$$

$$y(0) = 3 \longrightarrow k = \frac{3-2}{3+2} = \frac{1}{5}$$

$$\Rightarrow \boxed{y = \frac{2(5+e^{x^2})}{5-e^{x^2}}}$$

Problem 2 (15 points) Find the general solution of the following equation for $a > 0$ and show that all of its solutions tend to zero as $t \to \infty$.

$$y'' + 2a\,y' + 2a^2\,y = e^{-t}.$$

$$r^2 + 2ar + 2a^2 = 0 \Rightarrow \quad r = -a \pm \sqrt{a^2 - 2a^2} = a(-1 \pm i)$$

$$y_H(t) = \underbrace{c_1 e^{-at} \cos(at)}_{y_1} + \underbrace{c_2 e^{-at} \sin(at)}_{y_2}$$

$$y_p(t) = \int_0^t G(t,s) e^{-s}\, ds$$

$$G(t,s) = \frac{\begin{vmatrix} e^{-as} \sin(as) & e^{-as} \cos(as) \\ e^{-at} \sin(at) & e^{-at} \cos(at) \end{vmatrix}}{\begin{vmatrix} e^{-as} \cos as & e^{-as} \sin as \\ ae^{-as}(-\cos as - \sin as) & ae^{-as}(-\sin as + \cos as) \end{vmatrix}}$$

$$= \frac{e^{-a(s+t)} \left(\sin(as) \cos(at) - \cos(as) \sin(at) \right)}{a e^{-2as} \left(-\sin as \cos as + \cos^2 as + \sin as \cos as + \sin^2(as) \right)}$$

$$= \frac{e^{-at} e^{as}}{a} \left[\cos(at) \sin as - \sin(at) \cos(as) \right]$$

$$y_p(t) = \frac{e^{-at}}{a} \left[\cos(at) \underbrace{\int_0^t e^{(a-1)s} \sin(as)\, ds}_{I_1(t)} - \sin(at) \underbrace{\int_0^t e^{(a-1)s} \cos(as)\, ds}_{I_2(t)} \right]$$

$$I_1(t) = \frac{e^{(a-1)s}}{a^2 + (a-1)^2} \left[-a \cos(as) + (a-1) \sin(as) \right] \Big|_0^t$$

$$= \frac{e^{(a-1)t} \left[-a \cos(at) + (a-1) \sin(at) \right] + a}{a^2 + (a-1)^2}$$

$$I_2(t) = \frac{e^{(a-1)s}}{a^2 + (a-1)^2} \left[a \sin(as) + (a-1) \cos(as) \right] \Big|_0^t$$

$$= \frac{e^{(a-1)t} \left[a \sin(at) + (a-1) \cos(at) \right] - (a-1)}{a^2 + (a-1)^2}$$

$$Y(t) = Y_H(t) + Y_p(t)$$

$$= c_1 e^{-at} \cos(at) + c_2 e^{-at} \sin(at) +$$

$$\frac{e^{-at}}{a} [\cos(at)\, I_1(t) - \sin(at)\, I_2(t)] \quad (0)$$

$$\lim_{t \to \infty} e^{-at} \cos(at)\, I_1(t) =$$

(1)

$$= \lim_{t \to \infty} \frac{e^{-t}[-a\cos(at) + (a-1)\sin(at)]\cos at - ae^{-at} a\, at}{a^2 + (a-1)^2} = 0$$

$$\lim_{t \to \infty} e^{-at} \sin(at)\, I_2(t) =$$

$$= \lim_{t \to \infty} \frac{e^{-t}[a\sin(at) + (a-1)\cos(at)]\sin(at) - (a-1)e^{-at}\sin(at)}{a^2 + (a-1)^2}$$

$$= 0 \quad (2)$$

$$\lim_{t \to \infty} e^{-at} \sin(at) = 0 = \lim_{t \to \infty} e^{-at} \cos(at) \quad (3)$$

$$(0) - (3) \Rightarrow \lim_{t \to \infty} Y(t) = 0 .$$

Problem 3a (10 points) Find the recurrence relation for the power series solution of the equation $y' - xy = e^x$ about $x = 0$.

$$y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Rightarrow \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$(m := n-1) \qquad (m := n+1) \qquad (m := n)$

$$\Rightarrow \sum_{m=-1}^{\infty} (m+1) a_{m+1} x^m - \sum_{m=1}^{\infty} a_{m-1} x^m = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

$$\Rightarrow 0 + a_1 + \sum_{m=1}^{\infty} (m+1) a_{m+1} x^m - \sum_{m=1}^{\infty} a_{m-1} x^m = 1 + \sum_{m=1}^{\infty} \frac{x^m}{m!}$$

$$\Rightarrow a_1 - 1 + \sum_{m=1}^{\infty} \left[(m+1) a_{m+1} - a_{m-1} - \frac{1}{m!} \right] x^m = 0$$

$$\Rightarrow \boxed{a_1 = 1} \quad \& \quad \forall m \geq 1 : (m+1) a_{m+1} - a_{m-1} - \frac{1}{m!} = 0$$

$$\Rightarrow \boxed{a_{m+1} = \frac{a_{m-1}}{m+1} + \frac{1}{(m+1)!}, \quad m \geq 1}$$

or

For $n := m-1$

$$\Rightarrow \boxed{a_{n+2} = \frac{a_n}{n+2} + \frac{1}{(n+2)!}, \quad n \geq 0}$$

Problem 3b (5 points) Determine the first four nonzero terms in the power series solution of the following initial-value problem about $x = 0$.

$$y' - xy = e^x, \qquad y(0) = 1.$$

$n = 0$: $a_2 = \frac{a_1}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1 \Rightarrow \boxed{a_2 = 1}$

$n = 1$: $a_3 = \frac{a_2}{3} + \frac{1}{3!} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2} \Rightarrow \boxed{a_3 = \frac{1}{2}}$

$n = 2$: $a_4 = \frac{a_3}{4} + \frac{1}{4!} = \frac{1}{8} + \frac{1}{24} = \frac{1}{6} \Rightarrow \boxed{a_4 = \frac{1}{6}}$

$$\Rightarrow Y(x) = a_0 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \dots \quad Y(0) = 1 \Rightarrow \boxed{a_0 = 1}$$

$$\Rightarrow \boxed{Y(x) = 1 + x + x^2 + \frac{x^3}{2} + \dots}$$

Problem 4 (10 points) Let a and b be positive real numbers, $f(t)$ be a function with Laplace transform $F(s) := \mathcal{L}\{f(t)\}$ for $s > a$, and $u_b(t)$ be the unit step function. Show that

$t-b$

$$\mathcal{L}\{u_b(t)f(t)\} = e^{-bs}F(s), \qquad s > a.$$

$$\mathcal{L}\{u_b(t)\, f(t-b)\} = \int_0^\infty e^{-st}\, u_b(t)\, f(t-b)\, dt$$

$$= \int_b^\infty e^{-st} f(t-b)\, dt \qquad \text{let} \quad t-b =: \tau, \quad dt = d\tau$$

$$= \int_0^\infty e^{-s(b+\tau)} f(\tau)\, d\tau$$

$$= e^{-sb} \int_0^\infty e^{-s\tau} f(\tau)\, d\tau$$

for $s > a$.

$$= e^{-sb} F(s)$$

Problem 5 (20 points) Solve the following initial-value problem.

$$x_1'(t) = -3x_1(t) + x_2(t) - 1$$
$$x_2'(t) = 2x_1(t) - 4x_2(t) - 2$$
$$x_1(0) = 1, \qquad x_2(0) = 0.$$

$\vec{X} = \mathbb{A}\vec{X} + \vec{g}$, $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\quad \mathbb{A} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix}$, $\quad \vec{g} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

$\vec{x} = e^{rt}\vec{\xi} \qquad \begin{bmatrix} -3-\lambda & 1 \\ 2 & -4-\lambda \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(\lambda+3)(\lambda+4) - 2 = 0 \Rightarrow \lambda^2 + 7\lambda + 10 = 0$

$\lambda = \dfrac{-7 \pm \sqrt{49-40}}{2} = \dfrac{-7 \pm 3}{2} = \begin{cases} \lambda_1 = -5 \\ \lambda = -2 \end{cases}$

$\underline{\lambda_1 = -5} \Rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2\xi_1 = \xi_2$

$\vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \vec{X}^{(1)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-5t}$

$\underline{\lambda_2 = -2} \Rightarrow \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -\xi_1 + \xi_2 = 0$

$\vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \vec{X}^{(2)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$

$\Psi(t) = \begin{bmatrix} e^{-5t} & e^{-2t} \\ 2e^{-5t} & e^{-2t} \end{bmatrix}$

$\vec{X}(t) = \Psi(t)\vec{C} + \Psi(t)\displaystyle\int_0^t \Psi^{-1}(s)\,\vec{g}(s)\,ds$

$\Psi^{-1}(t) = \dfrac{1}{e^{-7t} - 2e^{-7t}}\begin{bmatrix} e^{-2t} & -e^{-2t} \\ -2e^{-5t} & e^{-5t} \end{bmatrix} = \begin{bmatrix} -e^{5t} & e^{5t} \\ 2e^{2t} & -e^{2t} \end{bmatrix}$

$\Psi^{-1}(s)\,\vec{g}(s) = \begin{bmatrix} -e^{5s} & e^{5s} \\ 2e^{2s} & -e^{2s} \end{bmatrix}\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -e^{5s} \\ 0 \end{bmatrix}$

$$\int_0^t \Psi^{-1}(s)\vec{g}(s)\,ds = \begin{bmatrix} \frac{-e^{5s}}{5}\Big|_0^t \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1-e^{5t}}{5} \\ 0 \end{bmatrix}$$

$$\Rightarrow \Psi(t)\int_0^t \Psi^{-1}(s)\vec{g}(s)\,ds = \begin{bmatrix} e^{-5t} & e^{-2t} \\ 2e^{-5t} & e^{-2t} \end{bmatrix}\begin{bmatrix} \frac{1-e^{5t}}{5} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{-5t}-1}{5} \\ \frac{2}{5}(e^{-5t}-1) \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{X}(0) = \Psi(0)\vec{C} \Rightarrow \vec{C} = \Psi(0)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{C} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \Psi(t)\vec{C} = \begin{bmatrix} e^{-5t} & e^{-2t} \\ 2e^{-5t} & e^{-2t} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -e^{-5t}+2e^{-2t} \\ -2e^{-5t}+2e^{-2t} \end{bmatrix}$$

$$\Rightarrow \vec{X}(t) = \begin{bmatrix} -e^{-5t} + 2e^{-2t} + \frac{e^{-5t}-1}{5} \\ -2e^{-5t} + 2e^{-2t} + \frac{2}{5}(e^{-5t}-1) \end{bmatrix}$$

$$x_1 = -\frac{4}{5}e^{-5t} + 2e^{-2t} - \frac{1}{5}$$

$$x_2 = -\frac{8}{5}e^{-5t} + 2e^{-2t} - \frac{2}{5}$$

Problem 6 (15 points) Find the Fourier series for the function $f(x)$ which is periodic with period 2π and satisfies

$$f(x) = \begin{cases} -1 & \text{for} \quad -\pi < x < 0, \\ 2 & \text{for} \quad 0 \le x \le \pi. \end{cases}$$

Simplify your response as much as possible.

Fourier series $\tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

$$a_n = \frac{1}{\pi} \left[\int_{-\pi}^{0} -\cos(nx)\, dx + \int_0^{\pi} 2\cos(nx)\, dx \right]$$

$$\Rightarrow \quad a_0 = \frac{1}{\pi} \left(-x \Big|_{-\pi}^{0} + 2x \Big|_0^{\pi} \right) = \frac{1}{\pi}(-\pi + 2\pi) = 1$$

$$n \geq 1: \quad a_n = \frac{1}{\pi} \left[-\frac{\sin(nx)}{n} \Big|_{\pi}^{0} + 2\frac{\sin(nx)}{n} \Big|_0^{\pi} \right] = 0$$

$$b_n = \frac{1}{\pi} \left[\int_{-\pi}^{0} -\sin(nx)\, dx + \int_0^{\pi} 2\sin(nx)\, dx \right]$$

$$= \frac{1}{\pi} \left[\frac{\cos(nx)}{n} \Big|_{-\pi}^{0} - \frac{2\cos(nx)}{n} \Big|_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{1 - \cos(n\pi)}{n} - \frac{2(\cos(n\pi) - 1)}{n} \right]$$

$$= \frac{3}{\pi n} [1 - \cos(n\pi)] = \frac{3[1 - (-1)^n]}{\pi n}$$

$$\Rightarrow \quad b_{2n} = 0, \qquad b_{2n-1} = \frac{6}{\pi(2n-1)} \qquad n \geq 1$$

$$\Rightarrow \quad \tilde{f}(x) = 1 + \sum_{n=1}^{\infty} \frac{6 \sin[(2n-1)x]}{\pi(2n-1)}$$

$$= 1 + \frac{6}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)x]}{2n-1}$$

Problem 7a (20 points) Use the method of separation of variables to solve the following problem.

$$u_t = u_{xx} + u, \quad x \in (0, \pi), \ t > 0$$
$$u(0,t) = 0, \quad u(\pi, t) = 0, \ t > 0$$
$$u(x, 0) = 1, \quad x \in (0, \pi).$$

$$u(x,t) = X(x)T(t) \Rightarrow T'X = TX'' + TX$$

$$\Rightarrow \frac{T'}{T} = \frac{X''}{X} + 1 \Rightarrow \frac{T'}{T} - 1 = \frac{X''}{X} = \lambda$$

$$\frac{T'}{T} = \lambda + 1 \Rightarrow \quad T(t) = c_1 e^{(\lambda+1)t}$$

$$\frac{X''}{X} = \lambda \quad X(0) = X(\pi) = 0 \Rightarrow \quad \lambda = -n^2, \ n = 1, 2, \ldots$$

$$X(x) = c_n \sin(nx)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n e^{(-n^2+1)t} \sin(nx)$$

For $t=0$

$$1 = \sum_{n=1}^{\infty} b_n \sin(nx) \Rightarrow b_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx)\, dx$$

$$\Rightarrow b_n = \frac{2}{\pi} \left(\frac{-\cos(nx)}{n} \right) \Big|_0^{\pi} = \frac{2(1 - \cos(n\pi))}{\pi n} = \frac{2[1-(-1)^n]}{\pi n}$$

$$\Rightarrow b_{2m} = 0$$

$$b_{2m-1} = \frac{4}{\pi(2m-1)} \quad m \geq 1$$

$$\Rightarrow u(x,t) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{e^{[-(2m-1)^2+1]t} \sin[(2m-1)x]}{2m-1}$$

Problem 7b (5 points) Determine the value of $u(x,t)$ at $x = \frac{\pi}{2}$ as $t \to \infty$, i.e., find $\lim_{t\to\infty} u(\frac{\pi}{2}, t)$.

$$\lim_{t\to\infty} u(\tfrac{\pi}{2}, t) = \frac{4 \sin(\frac{\pi}{2})}{\pi} = \frac{4}{\pi}.$$