Sol'ns

Math 204: Midterm Exam # 1
Spring 2014

- Write your name and Student ID number in the space provided below and sign.

Name and Last Name:	
ID Number:	
Signature:	

- Specify the section you have registered for by marking one of the boxes provided below.

  ☐ Section 1 (Mon. & Wed. 9:30-10:45, Instructor: Varga Kalantarov)

  ☐ Section 2 (Mon. & Wed. 15:30-16:45, Instructor: Ali Mostafazadeh)

- You have 90 minutes.
- You must show all your work. You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.

To be filled by the grader:

Problem 1:	
Problem 2:	
Problem 3:	
Problem 4:	
Problem 5:	
Total Grade:	

Problem 1 (15 points) Let f be a continuous function such that $f(t) \to 0$ as $t \to \infty$, and ϕ be the solution of the initial-value problem

$$y'(t) + 2y(t) = f(t), \qquad y(0) = 1.$$

Show that $\phi(t) \to 0$ as $t \to \infty$.

$\mu = e^{2t}$ $\quad (\mu y)' = e^{2t}(y' + 2y) = e^{2t} f$

$\Rightarrow y(t) = e^{-2t} \int e^{2t} f(t)\, dt$

$= e^{-2t} \left[\int_0^t e^{2s} f(s)\, ds + c \right]$

$y(0) = 1 \quad \Rightarrow \quad c = 1$

$\Rightarrow \quad \phi(t) = e^{-2t} + e^{-2t} \int_0^t f(s)\, ds$

$\lim_{t\to\infty} \phi(t) = \lim_{t\to\infty} \overset{\to 0}{e^{-2t}} + \lim_{t\to\infty} \dfrac{\int_0^t f(s)\, ds}{e^{2t}}$

If $\lim_{t\to\infty} \int_0^t f(s)\, ds < \infty \Rightarrow \lim_{t\to\infty} \phi(t) = 0$

If $\lim_{t\to\infty} \int_0^t f(s)\, ds = \infty \Rightarrow$

$\lim_{t\to\infty} \phi(t) = \lim_{t\to\infty} \dfrac{\frac{d}{dt}\int_0^t f(s)\, ds}{\frac{d}{dt} e^{2t}} = \lim_{t\to\infty} \dfrac{f(t)}{2e^{2t}}$

$= \left(\lim_{t\to\infty} \dfrac{e^{-2t}}{2}\right)\left(\lim_{t\to\infty} f(t)\right) = 0$

$\quad 0 \qquad\qquad 0$

Problem 2 (25 points) Show that the equation

$$(3x^2 \cos y - \sin y) \cos y \, dx - x \, dy = 0$$

has an integrating factor depending on y, and find the solution of this equation that satisfies the initial condition $y(1) = 0$.

$M := 3x^2 \cos^2 y - \sin y \cos y \qquad N := -x$

$M_y = -6x^2 \sin y \cos y - \cos^2 y + \sin^2 y \qquad N_x = -1$

$(\mu M)_y - (\mu N)_x = 0$

$\Rightarrow \quad M \mu_y - N \mu_x = \mu (N_x - M_y)$

$\Rightarrow \quad \left(\dfrac{M}{N_x - M_y}\right) \dfrac{\mu_y}{\mu} - \left(\dfrac{N}{N_x - M_y}\right) \dfrac{\mu_x}{\mu} = 1$

$$\frac{N_x - M_y}{M} = \frac{-1 + 6x^2 \sin y \cos y + \cos^2 y - \sin^2 y}{3x^2 \cos^2 y - \sin y \cos y}$$

$$= \frac{6x^2 \sin y \cos y - 2\sin^2 y}{(3x^2 \cos y - \sin y) \cos y} = \frac{2(3x^2 \cos y - \sin y) \sin y}{(3x^2 \cos y - \sin y) \cos y}$$

$= 2 \tan y \qquad$ can take $\quad \mu = \mu(y)$

$\Rightarrow \quad \dfrac{1}{2 \tan y} \dfrac{\mu'}{\mu} = 1 \Rightarrow \dfrac{\mu'}{\mu} = 2 \tan y$

$\Rightarrow \quad \ln \mu = 2 \displaystyle\int \tan y \, dy = 2 \int \frac{\sin y \, dy}{\cos y} = 2 \int \frac{-d(\cos y)}{\cos y} dy$

$= -2 \ln |\cos y| + c$
↳ take $c = 0$

$\Rightarrow \quad \mu = (\cos y)^{-2}$

$\tilde{M} = \mu M = (3x^2 \cos y - \sin y)(\cos y)^{-1} = 3x^2 - \tan y$

$\tilde{N} = \mu N = -x (\cos y)^{-2} = -x(1 + \tan^2 y)$

$$(3x^2 - \tan y)dx - \frac{x}{\cos^2 y}dy = 0$$

$$\phi_x = 3x^2 - \tan y, \quad \phi_y = -\frac{x}{\cos^2 y}.$$

$$\phi(x,y) = \int (3x^2 - \tan y)dx = x^3 - x\tan y + g(y),$$

$$(x^3 - x\tan y + g(y))_y = -\frac{x}{\cos^2 y},$$

$$-\frac{x}{\cos^2 y} + g'(y) = -\frac{x}{\cos^2 y} \Rightarrow g'(y) = 0 \Rightarrow g = C.$$

Hence $\phi(x,y) = x^3 - x\tan y + C,$

$$x^3 - x\tan y = C.$$

$y(1) = 0 \Rightarrow \quad 1^3 - 1\tan 0 = C \Rightarrow C = 1$

Solution: $\underline{x^3 - x\tan y = 1}.$

Problem 3 (15 points) Solve the following initial-value problem.

$$y'' - 6y' + 13y = 0, \qquad y(0) = 1, \qquad y'(0) = 7.$$

$r^2 - 6r + 13 = 0 \qquad r = 3 \pm \sqrt{9-13} = 3 \pm 2i$

$Y(t) = e^{3t}[C_1 \sin(2t) + C_2 \cos(2t)]$

$Y'(t) = e^{3t}[2C_1 \cos(2t) - 2C_2 \sin(2t) + 3C_1 \sin(2t) + 3C_2 \cos(2t)]$

$= e^{3t}[(3C_1 - 2C_2)\sin(2t) + (2C_1 + 3C_2)\cos(2t)]$

$Y(0) = 1 \Rightarrow \boxed{C_2 = 1}$

$Y'(0) = 7 \Rightarrow 2C_1 + 3 = 7$

$\boxed{C_1 = 2}$

$\Rightarrow \boxed{Y(t) = e^{3t}[2\sin(2t) + \cos(2t)]}$

Problem 4 (30 points) Given that $y_1(t) = t$ is a solution of the equation:

$$(t^2+3)y'' - 2ty' + 2y = 0,$$

on $\mathbb{R}$ solve the following initial-value problem.

$$(t^2+3)y'' - 2ty' + 2y = 0, \qquad y(0) = 3, \qquad y'(0) = 2.$$

$$y'' \underbrace{- \frac{2t}{t^2+3}}_{p} y' + \frac{2}{t^2+3} y = 0$$

$$y_1 y_2' - y_2 y_1' = c\, e^{-\int p\,dt} \quad \text{take} \quad c = 1$$

$$-\int p\,dt = \int \frac{2t}{t^2+3}\,dt = \ln(t^2+3)$$

$$\Rightarrow \quad t y_2' - y_2 = t^2 + 3 \quad \Rightarrow \quad y_2' - \frac{y_2}{t} = t + \frac{3}{t}$$

$$\mu = e^{\int -\frac{dt}{t}} = e^{-\ln t} = \frac{1}{t}$$

$$\Rightarrow \left(\frac{y_2}{t}\right)' = \frac{y_2'}{t} - \frac{y_2}{t^2} = \frac{1}{t}\left(t + \frac{3}{t}\right) = 1 + \frac{3}{t^2}$$

$$\Rightarrow \quad y_2 = t \int \left(1 + \frac{3}{t^2}\right) dt = t\left(t - \frac{3}{t} + c'\right)$$

$$= t^2 - 3 + c't$$

↳ take $c' = 0$

$\Rightarrow \quad y_2 = t^2 - 3$ is also a solution.

$$\Rightarrow \quad y(t) = c_1 t + c_2 (t^2 - 3)$$

$$y'(t) = c_1 + 2c_2 t$$

$y(0) = 3 \quad$ ↳ $\quad -3c_2 = 3 \quad \Rightarrow \quad \boxed{c_2 = -1}$

$y'(0) = 2 \quad \Rightarrow \quad \boxed{c_1 = 2}$

$$\Rightarrow \quad y(t) = 2t - (t^2 - 3) = -t^2 + 2t + 3$$

Problem 5 (15 points) Let p and q be continuous functions on $\mathbb{R}$. Find nonzero solutions y_1 and y_2 of the homogeneous equation $y'' + p(t)y' + q(t)y = 0$ on $\mathbb{R}$ such that their Wronskian is equal to $y_1 y_2$, i.e., $W[y_1(t), y_2(t)] = y_1(t)y_2(t)$.

$$y_1 y_2' - y_2 y_1' = y_1 y_2$$

$$\Rightarrow \frac{y_2'}{y_2} - \frac{y_1'}{y_1} = 1 \Rightarrow \frac{d}{dt}\underbrace{(\ln y_2 - \ln y_1)}_{\ln \frac{y_2}{y_1}} = 1$$

$$\Rightarrow \ln \frac{y_2}{y_1} = t + k \Rightarrow \boxed{y_2 = c_1 e^{t} y_1}$$

$$y_1 y_2 = W[y_1, y_2] = c_2 e^{-\int p(t)\,dt}$$

$$\Downarrow$$

$$c_1 e^{t} y_1^2 = c_2 e^{-\int p(t)\,dt}$$

$$\Rightarrow y_1^2 = \frac{c_2}{c_1} e^{-t-\int p(t)\,dt} \qquad \text{let } a_1 := \pm\sqrt{\frac{c_2}{c_1}}$$

$$a_2 := \pm\sqrt{c_1 c_2}$$

$$\Rightarrow y_1 = a_1 e^{-\frac{1}{2}\left[t + \int p(t)\,dt\right]}$$

$$\Rightarrow y_2 = a_2 e^{-\frac{1}{2}\left[-t + \int p(t)\,dt\right]}$$