Solution

Math 207: Midterm Exam # 2A

Fall 2004

- Write your name and Student ID number in the space provided below and sign.

Student's Name:	
ID Number:	
Signature:	

- You have 80 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the exam within the first 10 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)
- (Optional) Grade your own work out of 100. Record your estimated grade here:

Estimated Grade:	

If your expected grade and actual grade will turn out to differ by 9 points or less, you will be given the highest of the two.

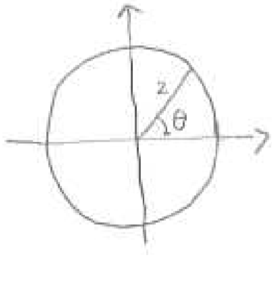
To be filled by the grader:

Actual Grade:	
Adjusted Grade:	

1. Let $C = \{z \in \mathbb{C} \mid |z| = 2\}$ and $f : \mathbb{C} \to \mathbb{C}$ be a function that is analytic in $\mathbb{C} - \{0\}$ (everywhere in $\mathbb{C}$ except $z = 0$).

1.a) For any ~~positive~~ integer n, evaluate the following contour integral without using the Residue Theorem, i.e., view and evaluate it as a line integral in $\mathbb{R}^2$.

$$I_n = \oint_C \frac{dz}{z^n}. \qquad \text{(10 points)}$$



$z = 2e^{i\theta} \qquad \theta \in [0, 2\pi) \qquad dz = 2ie^{i\theta}d\theta$

$$I_n = \int_0^{2\pi} \frac{2ie^{i\theta}d\theta}{2^n e^{in\theta}} = i2^{1-n}\int_0^{2\pi} e^{i(1-n)\theta}d\theta$$

$$= \begin{cases} i\int_0^{2\pi} d\theta = 2\pi i & \text{for } n = 1 \\ i2^{1-n} \left.\dfrac{e^{i(1-n)\theta}}{i(1-n)}\right|_0^{2\pi} = \left(\dfrac{2^{1-n}}{1-n}\right)\left[\underbrace{e^{i(1-n)2\pi}}_{1} - 1\right] = 0 & \text{for } n \neq 1 \end{cases}$$

$\Rightarrow \quad I_n = 2\pi i\,\delta_{1,n} \quad$ for all $n \in \mathbb{Z}$

1.b) Use your response for 1.a) to prove the Residue Theorem for f and C, i.e., show

$$\oint_C f(z)\,dz = 2\pi i\,R(0),$$

where $R(0)$ denotes the residue of f at $z = 0$. (15 points)

$f(z)$ admits a Laurent series expansion along C $\Rightarrow$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{m=1}^{\infty} \frac{b_m}{z^m}$$

$$\Rightarrow \oint_C f(z)\,dz = \sum_{n=0}^{\infty} a_n \underbrace{\oint_C z^n dz}_{} + \sum_{m=1}^{\infty} b_m \underbrace{\oint_C \frac{dz}{z^m}}_{2\pi i\,\delta_{1,m}}$$

$$\underbrace{\oint_C \frac{dz}{z^{-n}} = 2\pi i\,\delta_{1,-n}}_{\text{because } -n \neq 1 \text{ for all } n \geq 0}$$

$$= 0 + 2\pi i\,b_1$$

But by definition $b_1 = R(0)$

$$\Rightarrow \oint_C f(z)\,dz = 2\pi i\,R(0).$$

2. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function at $z = 0$ and $g : \mathbb{C} \to \mathbb{C}$ be defined by $g(z) = \dfrac{f(z)}{z^n}$ for some positive integer n. Show that the residue $R(0)$ of g at $z = 0$ is given by $\dfrac{f^{(n-1)}(0)}{(n-1)!}$ where $f^{(k)}$ denotes the k-th derivative of f. (25 points)

Because f is analytic at $z = 0$ $\Rightarrow$

$$f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} z^m$$

$$\Rightarrow \quad g(z) = \sum_{n=0}^{\infty} \frac{f^{(m)}(0)}{m!} z^{m-n}$$

$$= f(0) z^{-n} + f'(0) z^{1-n} + \cdots + \frac{f^{(n-2)}(0)}{(n-2)!} z^{-2}$$

$$+ \frac{f^{(n-1)}(0)}{(n-1)!} z^{-1} + \frac{f^{(n)}(0)}{n!} + \frac{f^{(n+1)}(0)}{(n+1)!} z + \cdots$$

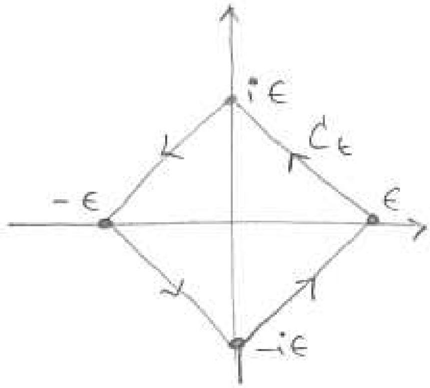
So $R(0) = b_1 = \dfrac{f^{(n-1)}(0)}{(n-1)!}$

Coefficient of $\frac{1}{z} = z^{-1}$.

3. Calculate

$$\lim_{\epsilon\to 0}\oint_{C_\epsilon}\frac{dz}{z(1-e^{-2z})},$$

where C_ϵ is the contour shown in the following figure. (25 points)



In the limit $\epsilon \to 0$, $z(1-e^{-2z})$ has a single zero - because this fn is analytic $\Rightarrow$ it has isolated zeros.

This implies that for sufficiently small ϵ, there is only one singular point of $\dfrac{1}{z(1-e^{-2z})}$.

$$z(1-e^{-2z}) = z\left[1-\left(1-2z+\frac{(2z)^2}{2!}-\frac{(2z)^3}{3!}+\cdots\right)\right]$$

$$= z\left(2z-\frac{4z^2}{2!}+\frac{8z^3}{3!}-\frac{16z^4}{4!}\pm\cdots\right)$$

$$= 2z^2\left(1-\frac{2z}{2!}+\frac{4z^2}{3!}-\frac{8z^3}{4!}+\cdots\right)$$

$\Rightarrow \lim_{z\to 0} z^2\left[\dfrac{1}{z(1-e^{-2z})}\right] = \dfrac{1}{2} \Rightarrow$ $z=0$ is a pole of order 2.

$$f(z) = \frac{1}{z(1-e^{-2z})} = \frac{g(z)}{z^2} \Rightarrow g(z) = \frac{1}{2\left(1-\frac{2z}{2!}+\frac{4z^2}{3!}\pm\cdots\right)}$$

is analytic at $z=0$

$\Rightarrow R(0) = g'(0)$

$$g'(z) = \frac{-\left(-1+\frac{8z}{3!}\pm\cdots\right)}{2\left(1-\frac{2z}{2!}+\frac{4z^2}{3!}\pm\cdots\right)^2} \Rightarrow R(0) = g'(0) = \frac{1}{2}$$

By residue thm $\displaystyle\lim_{\epsilon\to 0}\oint_{C_\epsilon}\frac{dz}{z(1-e^{-2z})} = 2\pi i R(0) = \pi i.$

4. Let $f(t)$ be the inverse Fourier transform of

$$\bar{f}(s) = \frac{\sin s}{s+1}.$$

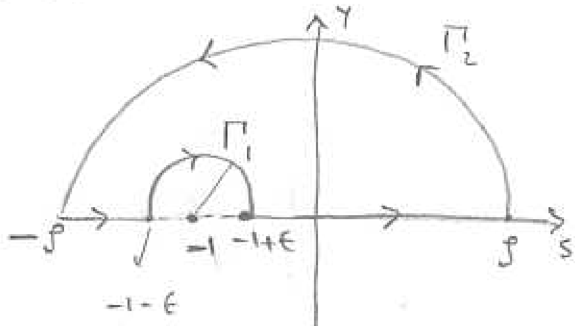
Use contour integration to determine $f(t)$ for all $t > +1$. (35 points)

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ist}\frac{\sin s}{s+1}\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ist}\left(e^{is}-e^{-is}\right)}{2i(s+1)}\,ds$$

$$= \frac{1}{4\pi i}\Big[\underbrace{\int_{-\infty}^{\infty}\frac{e^{i(t+1)s}}{s+1}\,ds}_{I_1} - \underbrace{\int_{-\infty}^{\infty}\frac{e^{i(t-1)s}}{s+1}\,ds}_{I_2}\Big]$$

$t > 1 \Rightarrow t \pm 1 > 0$

let $I_k = \int_{-\infty}^{\infty}\frac{e^{iks}}{s+1}\,ds$ for some $k \in \mathbb{R}^+$



$$\oint_C \frac{e^{ikz}}{z+1}\,dz = \int_{-\rho}^{-1-\epsilon}\frac{e^{iks}}{s+1}\,ds + \int_{-\Gamma_1}\frac{e^{ikz}}{z+1}\,dz + \int_{-1+\epsilon}^{\rho}\frac{e^{iks}}{s+1}\,ds + \int_{\Gamma_2}\frac{e^{iks}}{z+1}\,dz$$

$$\Rightarrow I_k = \lim_{\substack{\epsilon\to 0\\ \rho\to\infty}}\left[\oint_C \frac{e^{ikz}}{z+1}\,dz - \int_{-\Gamma_1}\frac{e^{ikz}}{z+1}\,dz - \int_{\Gamma_2}\frac{e^{ikz}}{z+1}\,dz\right]$$

Γ_2: $z = \rho e^{i\theta}$ $\theta \in (0,\pi) \Rightarrow \sin\theta > 0 \Rightarrow$

$$e^{ikz} = e^{ik\rho[\cos\theta + i\sin\theta]} = e^{-k\rho\sin\theta}\,e^{ik\rho\cos\theta}$$

in the limit $\rho\to\infty$, $e^{ikz}\to 0 \Rightarrow \int_{\Gamma_2}\frac{e^{ikz}}{z+1}\,dz = 0$

Γ_1: $z+1 = \epsilon e^{i\theta}$ $\theta\in(0,\pi) \Rightarrow dz = \epsilon i e^{i\theta}d\theta$

$$\lim_{\epsilon\to 0}\int_{\Gamma_1}\frac{e^{ikz}}{z+1}\,dz = \lim_{\epsilon\to 0}\int_0^{\pi}\frac{e^{ik[\epsilon e^{i\theta}-1]}}{\epsilon e^{i\theta}}\,i\epsilon e^{i\theta}\,d\theta = ie^{-ik}\int_0^{\pi}d\theta = \pi i e^{-ik}$$

$\oint_C \frac{e^{ikz}}{z+1}\,dz = 0$ because there are no singular points inside C.

$$\Rightarrow \quad \underline{I_k = 0 - (-i\pi e^{-ik}) = i\pi e^{-ik}}$$

$$\Rightarrow \quad I_1 = i\pi e^{-i(t+1)} \quad , \quad I_2 = i\pi e^{-i(t-1)}$$

$$\Rightarrow \quad f(t) = \frac{1}{4\pi i}\left(i\pi e^{-i(t+1)} - i\pi e^{-i(t-1)} \right)$$

$$= \frac{1}{4} e^{-it} (e^{-i} - e^{i})$$

$$= \frac{i e^{-it}}{2} \left(\frac{e^{-i} - e^{i}}{2i} \right)$$

$$= \frac{i e^{-it}}{2} \sin(-1)$$

$$\Rightarrow \quad \boxed{f(t) = - \frac{i \sin(1) e^{-it}}{2} .}$$