Name: Student ID:

Math 207: Quiz # 2A

Fall 2004

- You have 25 minutes.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 2 points will be deduced from your grade.

1. Let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{H}$ and $\mathbb{U}$ be $n \times n$ matrices such that $\mathbb{H}$ is Hermitian and $\mathbb{U}$ is unitary.

   1.a) Show that $(AB)^T = B^T A^T$, where T stands for the "transpose".
   Hint: Let $\mathbb{M} = \mathbb{A}\mathbb{B}$ and compute the entries of $\mathbb{M}^T$ in terms of the entries of $\mathbb{A}$ and $\mathbb{B}$. (3 points)

$$(\mathbb{M}^T)_{ij} = M_{ji} = (\mathbb{A}\mathbb{B})_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki} \qquad (1)$$

$$(\mathbb{B}^T\mathbb{A}^T)_{ij} = \sum_{k=1}^{n} (\mathbb{B}^T)_{ik} (\mathbb{A}^T)_{kj} = \sum_{k=1}^{n} B_{ki} A_{jk}$$

$$= \sum_{k=1}^{n} A_{jk} B_{ki} \qquad (2)$$

$$\Rightarrow (\mathbb{M}^T)_{ij} = (\mathbb{B}^T\mathbb{A}^T)_{ij} \Rightarrow \quad (\mathbb{A}\mathbb{B})^T = \mathbb{M}^T = \mathbb{B}^T\mathbb{A}^T \quad \boxtimes$$

   1.b) Show that $(AB)^* = B^* A^*$, where $*$ stands for the "Hermitian conjugate". (2 points)

$$(\mathbb{A}\mathbb{B})^* = (\overline{\mathbb{A}\mathbb{B}})^T = (\overline{\mathbb{A}}\,\overline{\mathbb{B}})^T = \overline{\mathbb{B}}^T\,\overline{\mathbb{A}}^T = \mathbb{B}^*\mathbb{A}^* \quad \boxtimes$$

   1.c) Show that $\mathbb{U}^{-1}\mathbb{H}\mathbb{U}$ is Hermitian. (5 points)

$$(\mathbb{U}^{-1}\mathbb{H}\mathbb{U})^* = \mathbb{U}^* (\mathbb{U}^{-1}\mathbb{H})^* = \mathbb{U}^* \mathbb{H}^* (\mathbb{U}^{-1})^*$$

$$= \mathbb{U}^* \mathbb{H} (\mathbb{U}^{-1})^{-1} \qquad \left(\begin{array}{l}\mathbb{H} \text{ is Hermitian, } \mathbb{U} \text{ is unit} \Rightarrow \\ \mathbb{U}^{-1} \text{ is unit}\end{array}\right)$$

$$= \mathbb{U}^{-1}\mathbb{H}\mathbb{U}$$

$\boxtimes$

2. Let a, b, r, s be real numbers and $\mathbb{U} = \begin{pmatrix} re^{ia} & 2s\,e^{-ia} \\ re^{-ib} & -2s\,e^{ib} \end{pmatrix}$. How should a and b be related so that $\mathbb{U}$ is a unitary matrix? Use the condition that $\mathbb{U}$ is unitary to determine r and s. (10 points)

If $\mathbb{U}$ is unit. $\vec{v}_1 = \begin{pmatrix} re^{ia} \\ re^{-ib} \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2se^{-ia} \\ -2se^{ib} \end{pmatrix}$

must be orthonormal $\Rightarrow$

$$0 = \langle \vec{v}_1, \vec{v}_2 \rangle = \vec{v}_1^{\,*} \vec{v}_2 = (re^{-ia} \quad re^{ib}) \begin{pmatrix} 2s\,e^{-ia} \\ -2s\,e^{ib} \end{pmatrix}$$

$$= 2rs\,(e^{-2ia} - e^{2ib}) \qquad (1)$$

$$1 = \langle \vec{v}_1, \vec{v}_1 \rangle = r^2 + r^2 = 2r^2 \Rightarrow \boxed{r = \pm \frac{1}{\sqrt{2}}} \qquad (2)$$

$$1 = \langle \vec{v}_2, \vec{v}_2 \rangle = 4s^2 + 4s^2 = 8s^2 \Rightarrow \boxed{s = \pm \frac{1}{\sqrt{8}}} \qquad (3)$$

$\underline{(1)-(3)}$ $\Rightarrow$ $e^{2ib} = e^{-2ia} \Rightarrow e^{2i(a+b)} = 1$

$\Rightarrow$ $2(a+b) = 2\pi k$ for some $k \in \mathbb{Z}$.

$\Downarrow$

$\boxed{b = -a + \pi k}$ 〃 〃 〃

$\Downarrow$

$$\left(e^{ib} = e^{-ia} e^{\pi k} = \pm e^{-ia} \right)$$