Math 208: Midterm Exam 1

Spring 2013

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	SOLUTIONS

- You have 80 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the exam within the first 10 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)
- (Optional) Grade your own work out of 100. Record your estimated grade here:

Estimated Grade:	

If your expected grade and actual grade will turn out to differ by 9 points or less, you will be given the highest of the two.

To be filled by the grader:

Actual Grade:	
Adjusted Grade:	

Problem 2. (20 points) Let $a \in \mathbb{R}$ and $\{b_n\}$ be a sequence in the interval $(0, \infty)$ that converges to a. Show that $a \geq 0$.

Let $b_n \in (0, \infty)$ $\forall n \in \mathbb{N}$ and b_n be a convergent sequence. Let $\lim_{n \to \infty} b_n = a$.

Suppose that $a < 0$. Then for $\varepsilon = -\frac{a}{2}$

$\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |b_n - a| < \varepsilon = -\frac{a}{2}$

Then for $n \geq N$

$$|b_n - a| < -\frac{a}{2} \iff \frac{3a}{2} < b_n < \frac{a}{2}$$

Since $\frac{a}{2} < 0$, this implies that

$$b_n < 0 \text{ for } n \geq N$$

which is a contradiction.

Problem 3. (30 points) Let $f : D \to \mathbb{R}$ be a continuous function with a sequentially compact domain $D \subseteq \mathbb{R}$. Show that the range of f, i.e., $f(D)$, is bounded above.

Suppose that $f(D)$ is not bounded above. Then $\forall n \in \mathbb{N}$ $\exists$ $x_n \in D$ such that $f(x_n) > n$.

Since D is sequentially compact there exist a subsequence x_{n_k} of x_n which converges to some point in D. Let $\lim_{k \to \infty} x_{n_k} = a$.

Since $a \in D$ and f is continuous at a,

$$\lim_{k \to \infty} f(x_{n_k}) = f(a)$$

In particular $f(x_{n_k})$ is convergent and so is bounded.
However $f(x_{n_k}) > n_k$ for all $k \in \mathbb{N}$
which is a contradiction with $f(D)$ being bounded above.
Hence $f(D)$ is bounded above.

Problem 4 (20 points) Let $f : D \to \mathbb{R}$ be a bounded uniformly continuous function with domain D. Show that $f^2 : D \to \mathbb{R}$ is also uniformly continuous. Recall that $f^2(x) := f(x)^2$ for all $x \in D$.

Let x_n and y_n be a sequence in D such that

$$\lim_{n\to\infty} x_n - y_n = 0.$$

We will show that

$$\lim_{n\to\infty} f^2(x_n) - f^2(y_n) = 0$$

Note that

$$|f^2(x_n) - f^2(y_n)| = |f(x_n) - f(y_n)|\,|f(x_n) + f(y_n)|$$

Since f is bounded $\exists\, M \in \mathbb{R}^+$ s.t. $\forall n \in \mathbb{N}$

$$|f(x_n) + f(y_n)| < M.$$

Then $|f^2(x_n) - f^2(y_n)| < M\,|f(x_n) - f(y_n)|$

Since f is uniformly continuous, $\lim_{n\to\infty} |f(x_n) - f(y_n)| = 0$.

Hence by comparison lemma

$$\lim_{n\to\infty} |f^2(x_n) - f^2(y_n)| = 0.$$

Problem 5 (20 points) Let $f : \mathbb{Q} \to \mathbb{R}$ be a strictly increasing function with domain $\mathbb{Q}$. Show that f cannot be onto, i.e., $f(\mathbb{Q}) \neq \mathbb{R}$.

Hint: Use the properties of the inverse of f.

Let $f: \mathbb{Q} \to \mathbb{R}$ be a strictly increasing function.

Suppose that $f(\mathbb{Q}) = \mathbb{R}$. Then f is one-to-one and $f^{-1}: \mathbb{R} \to \mathbb{Q}$ exist. If f^{-1} were continuous, then since $\mathbb{R}$ is interval $f^{-1}(\mathbb{R}) = \mathbb{Q}$ would be an interval. Since $\mathbb{Q}$ is not an interval, f^{-1} is not continuous on $\mathbb{R}$. So $\exists\, x \in \mathbb{R}$ s.t. f^{-1} is not continuous at x. Then

$\exists\, \varepsilon > 0$ s.t. $\forall\, \delta > 0$ $\exists\, y \in (x-\delta, x+\delta)$ satisfying $|f^{-1}(x) - f^{-1}(y)| > \varepsilon$.

Taking $\delta = \frac{1}{n}$, this implies that

$\exists\, y_n \in (x - \frac{1}{n}, x + \frac{1}{n})$ s.t. $|f^{-1}(x) - f^{-1}(y_n)| > \varepsilon$.

We can assume WLOG that there exist a monotone increasing subsequence y_{n_k} of y_n. (Theorem 2.32)

Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $f^{-1}(\mathbb{R}) = \mathbb{Q}$ there exist $d \in \mathbb{R}$ s.t.

$f^{-1}(x) - \varepsilon < f^{-1}(d) < f^{-1}(x)$.

Then $d < x$ and $\forall k \in \mathbb{N}$ $d > y_{n_k}$

since $f^{-1}(x) - f^{-1}(y_{n_k}) > \varepsilon \iff f^{-1}(y_{n_k}) < f^{-1}(x) - \varepsilon$.

But $\lim_{k\to\infty} y_{n_k} = x$, $d > y_{n_k}$ $\forall k \in \mathbb{N}$ and $d < x$, is not possible. $\boxtimes$