Math 208, Spring 2013, Quiz # 4

You have 40 minutes.

Name, Last Name	Student ID Number	Signature

Problem 1 (10 points) Give the definition or precise statement of the following.

1.a) A limit point of a subset A of $\mathbb{R}^n$:

1.b) A continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$:

1.c) The directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ in the direction $\mathbf{p}$ at a point $\mathbf{x}$:

1.d) Mean-Value Theorem for a function $f : \mathcal{O} \to \mathbb{R}$ where $\mathcal{O}$ is an open subset of R^n:

Problem 2 (6 points) Let $n \in \mathbb{N}$ and $A \subseteq \mathbb{R}^n$ be such that every limit point of A belongs to A. Show that A is closed.

Let x_n be a convergent sequence in A

and let $x := \lim_{n\to\infty} x_n$

If $x_n = x$ for some n, then $x \in A$ (since $x_n \in A$ for all $n \in \mathbb{N}$) and we are done. If $x_n \neq x$ for all n then by definition x is a limit point of A. Since A contains limit points, $x \in A$ again. Hence A is closed.

Problem 3 (6 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined on $\mathbb{R}^2$ such that for all $(x, y) \in \mathbb{R}^2$, $|f(x, y)| \le x^2 + y^2$. Show that f has first order partial derivatives at $(0, 0)$.

We first show that

$$\lim_{t \to 0} \frac{f(0+t, 0) - f(0,0)}{t}$$

exist. Let t_n be a sequence of real numbers in $\mathbb{R}\setminus\{0\}$ converging to 0. Then

$$\left| \frac{f(0+t_n, 0) - f(0,0)}{t_n} \right| = \left| \frac{f(0+t_n, 0) - 0}{t_n} \right| = \left| \frac{f(t_n, 0)}{t_n} \right| \le \frac{t_n^2}{|t_n|} \le |t_n|$$

Since $|t_n| \to 0$, by comparison lemma,

$$\lim_{t \to 0} \frac{f(0+t, 0) - f(0,0)}{t} = 0$$

Hence f has first order partial derivative with respect to x at $(0,0)$.

Similarly, you can prove that f has partial derivative with respect to y.

Problem 4 (8 points) Let $n \in \mathbb{N}$, $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{p} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, and $\alpha \in \mathbb{R} \setminus \{0\}$. Show that

$$\frac{\partial f}{\partial(\alpha \mathbf{p})}(\mathbf{x}) = \alpha \frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}).$$

By the directional derivative theorem

$$\frac{\partial f}{\partial (\alpha p)}(x) = \sum_{i=1}^{n} \alpha p_i \frac{\partial f}{\partial x^i}(x) \quad \text{where } p = (p_1, \dots, p_n)$$

The last term equals to

$$\alpha \sum_{i=1}^{n} p_i \frac{\partial f}{\partial x^i}(x) = \alpha \frac{\partial f}{\partial p}(x)$$

which completes the proof