Name: Student ID: Signature:

Math 303: Quiz # 2

Fall 2004

- You have 30 minutes.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 2 points will be deduced from your grade.

1. Show that the following identity holds for all $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$. (5 points)

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = |\vec{a}|^2 (\vec{c} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})(\vec{a} \cdot \vec{c}).$$

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \sum_{i=1}^{3} (\vec{a} \times \vec{b})_i (\vec{c} \times \vec{d})_i = \sum_{i,j,k,l,m=1}^{3} \epsilon_{ijk} a_j b_k \epsilon_{ilm} c_l d_m$$

$$= \sum_{j,k,l,m=1}^{3} (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) a_j b_k c_l d_m$$

$$= \sum_{j,k=1}^{3} (a_j b_k c_j d_k - a_j b_k c_k d_j)$$

$$= (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$$

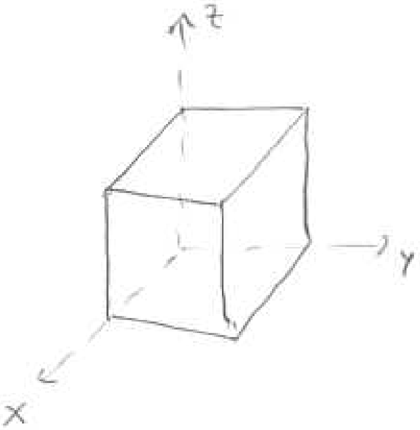
2. Let $\vec{F}(x, y, z) := xy\,\hat{i} + yz\,\hat{j} + xz\,\hat{k}$, where (x, y, z) are Cartesian coordinates and $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the x-, y-, and z-axes, respectively.

2.a) Is $\vec{F}$ a conservative force? Why? (5 points)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xy & yz & xz \end{vmatrix} = -y\,\hat{i} - z\,\hat{j} - x\,\hat{k} \neq 0$$

so $\vec{F} \neq -\vec{\nabla}\phi$ for any ϕ $\Rightarrow$ it is not conservative.

2.b) Give the statement of the Divergence theorem and use it to evaluate $\int_\sigma \vec{F} \cdot \hat{n}\, d\sigma$, where $\vec{F}(x,y,z) := xy\,\hat{\imath} + yz\,\hat{\jmath} + xz\,\hat{k}$, the surface σ is the boundary of the cube of unit side length that is shown in the following figure, and $\hat{n}$ is the unit outward normal vector to σ. (10 points)



$$I = \int_\sigma \vec{F} \cdot \hat{n}\, d\sigma = \int_V \vec{\nabla} \cdot \vec{F}\, dV$$

$$\vec{\nabla} \cdot \vec{F} = \partial_x (xy) + \partial_y (yz) + \partial_z (xz)$$

$$= y + z + x$$

$$I = \int_0^1 dx \int_0^1 dy \int_0^1 dz\, [x + y + z]$$

$$= 3 \int_0^1 u\, du \int_0^1 dv \int_0^1 dw$$

$$= 3 \left[\frac{u^2}{2} \Big|_0^1 + v \Big|_0^1 + w \Big|_0^1 \right] = 3 \left(\frac{1}{2} + 1 + 1 \right)$$

$$= \frac{15}{2}.$$