Solution

Name: Student ID: Signature:

Math 303: Quiz # 4

Fall 2004

- You have 40 minutes.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 2 points will be deduced from your grade.

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1. Let $f:\mathbb{R}\to\mathbb{C}$ be a square-integrable function with Fourier transform $\tilde{f}$. Show that $\int_{-\infty}^{\infty}|f(x)|^2dx = c\int_{-\infty}^{\infty}|\tilde{f}(k)|^2dk$, where c is a nonzero constant independent of f. (5 points) Hint: Use the bra-ket notation.

$$\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty} f^*(x)f(x)\,dx = \int_{-\infty}^{\infty}\langle x|f\rangle^*\langle x|f\rangle\,dx$$

$$= \int_{-\infty}^{\infty}\langle f|x\rangle\langle x|f\rangle\,dx = \langle f|f\rangle = \int_{-\infty}^{\infty}\langle f|k\rangle\langle k|f\rangle\,dk$$

$$= \int_{-\infty}^{\infty}\langle k|f\rangle^*\langle k|f\rangle\,dk = \int_{-\infty}^{\infty}\tilde{f}(k)^*\tilde{f}(k)\,dk = \int_{-\infty}^{\infty}|\tilde{f}(k)|dk$$

So $c=1$ in the convention I use.

2. Compute $\dfrac{\tilde{f}(2k)}{\tilde{f}(k)}$ where $\tilde{f}$ is the Fourier transform of the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{for } |x|\le 1\\ 0 & \text{for } |x|>1.\end{cases} \qquad \text{(7 points)}$$

$$\tilde{f}(k) = \langle k|f\rangle = \int_{-\infty}^{\infty}\langle k|x\rangle\langle x|f\rangle\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\left.\frac{e^{-ikx}}{-ik}\right|_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{e^{-ik}-e^{ik}}{-ik} = \frac{1}{\sqrt{2\pi}}\cdot\frac{2}{k}\left(\frac{e^{ik}-e^{-ik}}{2i}\right)$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\sin k}{k}$$

$$\Rightarrow \frac{\tilde{f}(2k)}{\tilde{f}(k)} = \frac{\dfrac{\sin(2k)}{2k}}{\dfrac{\sin k}{k}} = \cos k$$

3. Let

$$\mathcal{F}[y(x)] := \int_0^1 \frac{y(x)}{y'(x)}\,dx.$$

Find the most general twice differentiable function $y : [0,1] \to \mathbb{R}$ for which $\mathcal{F}[y(x)]$ exists and $\delta\mathcal{F}[y(x)] = 0$, alternatively $\dfrac{\delta \mathcal{F}[y(x)]}{\delta y(\tilde{x})} = 0.$ (8 points)

$$\delta \mathcal{F}[y(x)] = 0 \Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \quad \text{when} \quad F = \frac{y}{y'}$$

$$\frac{\partial F}{\partial y} = \frac{1}{y'}, \quad \frac{\partial F}{\partial y'} = -\frac{y}{y'^2}$$

$$\Rightarrow \frac{1}{y'} - \frac{d}{dx}\left(-\frac{y}{y'^2}\right) = 0$$

$$\Rightarrow \frac{1}{y'} + \left[\frac{y'^3 - 2y'y''y}{y'^4}\right] = 0$$

$$\Rightarrow \frac{1}{y'} + \frac{1}{y'} - \frac{2yy''}{y'^3} = 0$$

$$\Rightarrow 2\left(1 - \frac{yy''}{y'^2}\right) = 0$$

$$\Rightarrow \frac{y'^2 - yy''}{y'^2} = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{y'}\right) = \frac{y'^2 - y''y}{y'^2} = 0 \Rightarrow$$

$$\frac{y}{y'} = c \Rightarrow \frac{y'}{y} = \frac{1}{c} \Rightarrow \frac{d}{dx}\ln y = \frac{1}{c}$$

$$\Rightarrow \ln y = \frac{x}{c} + k \Rightarrow y = e^{\frac{x}{c} + k}$$

$$\Rightarrow \boxed{y = a e^{bx}} \quad \text{when} \quad a = e^k \ \& \ b = \frac{1}{c}$$

are constants.