From: "Mathematical Physics," by B. Kusse & E. Westwig (John-Wiley, 1998)

5

THE DIRAC δ-FUNCTION

The Dirac δ-function is a strange, but useful function which has many applications in science, engineering, and mathematics. The δ-function was proposed in 1930 by Paul Dirac in the development of the mathematical formalism of quantum mechanics. He required a function which was zero everywhere, except at a single point, where it was discontinuous and behaved like an infinitely high, infinitely narrow spike of unit area. Mathematicians were quick to point out that, strictly speaking, there is no function which has these properties. But Dirac supposed there was, and proceeded to use it so successfully that a new branch of mathematics was developed in order to justify its use. This area of mathematics is called the theory of *generalized functions* and develops, in complete detail, the foundation for the Dirac δ-function. This rigorous treatment is necessary to justify the use of these discontinuous functions, but for the physicist the simpler physical interpretations are just as important. We will take both approaches in this chapter.

5.1 EXAMPLES OF SINGULAR FUNCTIONS IN PHYSICS

Physical situations are usually described using equations and operations on continuous functions. Sometimes, however, it is useful to consider discontinuous idealizations, such as the mass density of a point mass, or the force of an infinitely fast mechanical impulse. The functions that describe these ideas are obviously extremely discontinuous, because they and all their derivatives must diverge. For this reason they are often called *singular* functions. The Dirac δ-function was developed to describe functions that involve these types of discontinuities and provide a method for handling them in equations which normally involve only continuous functions.

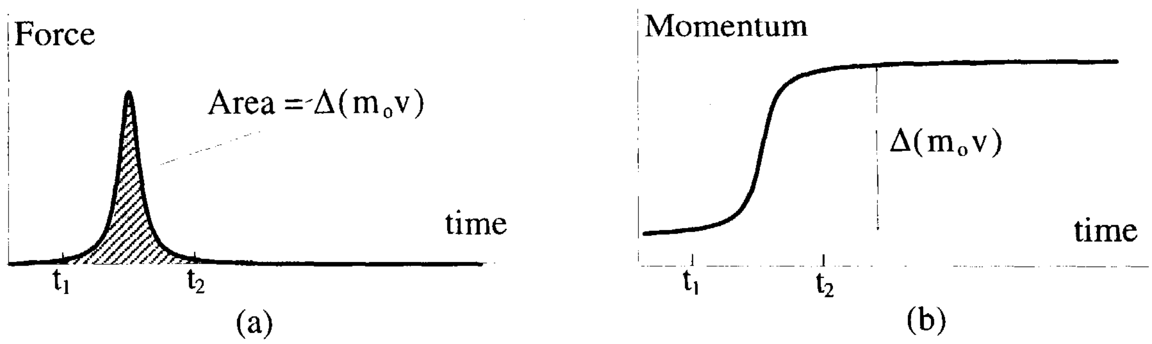


Figure 5.1 Force and Change in Momentum

5.1.1 The Ideal Impulse

Often a student's first encounter with the δ-function is the "ideal" impulse. In mechanics, an impulse is a force which acts on an object over a finite period of time. Consider the realistic force depicted in Figure 5.1(a). It is zero until $t = t_1$, when it increases smoothly from zero to its peak value, and then finally returns back to zero at $t = t_2$. When this force is applied to an object of mass m_o, the momentum in the direction of the applied force changes, as shown in Figure 5.1(b). The momentum remains constant until $t = t_1$, when it begins to change continuously until reaching its final value at $t = t_2$. The net momentum change $\Delta(m_o v)$ is equal to the integrated area of the force curve:

$$\begin{aligned}\int_{-\infty}^{+\infty} dt\, F(t) &= \int_{t_1}^{t_2} dt\, F(t) \\ &= \int_{t_1}^{t_2} dt\, m_o \frac{dv}{dt} = m_o v\Big|_{t_2} - m_o v\Big|_{t_1} = \Delta(m_o v).\end{aligned} \tag{5.1}$$

An ideal impulse produces all of its momentum change instantaneously, at the single point $t = t_o$, as shown in Figure 5.2(a). Of course this is not very realistic, since it requires an infinite force to change the momentum of a finite mass in zero time. But it is an acceptable thought experiment, because we might be considering the limit in which a physical process occurs faster than any measurement can detect.

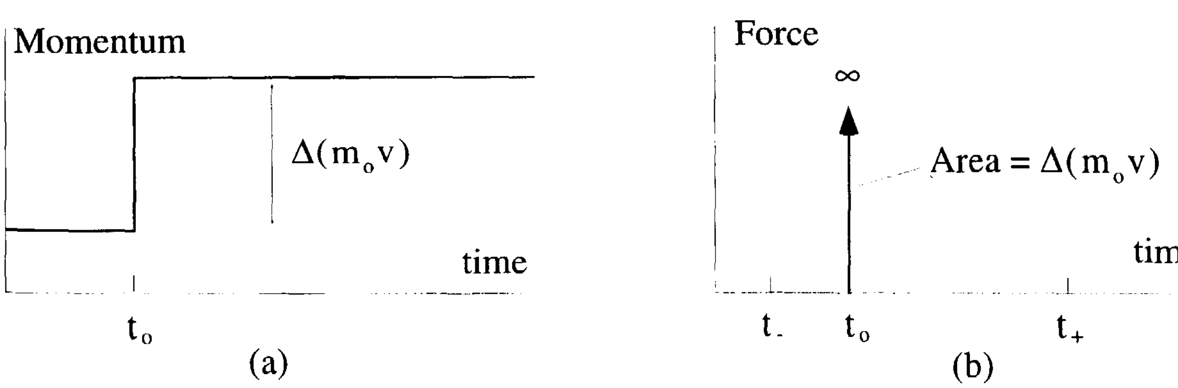


Figure 5.2 An Instantaneous Change in Momentum

The force of an ideal impulse cannot be graphed as a function of time in the normal sense. The force exists for only a single instant, and so is zero everywhere, except at $t = t_o$, when it is infinite. But this is not just any infinity. Since the total momentum change must be $\Delta(m_o v)$, the force must diverge so that its integral area obeys (for all $t_- < t_+$)

$$\int_{t_-}^{t_+} dt\, F(t) = \begin{cases} \Delta(m_o v) & t_- < t_o < t_+ \\ 0 & \text{otherwise} \end{cases} . \tag{5.2}$$

In other words, any integral which includes the point t_o gives a momentum change of $\Delta(m_o v)$. On the other hand, integrals which exclude t_o must give no momentum change. We graph this symbolically as shown in Figure 5.2(b). A spike of zero width with an arrow indicates the function goes to infinity, while the area of the impulse is usually indicated by a comment on the graph, as shown in the figure, or sometimes by the height of the arrow.

The Dirac δ-function $\delta(t)$ was designed to represent exactly this kind of "pathological" function. $\delta(t)$ is zero everywhere, except at $t = 0$, when it is infinite. Again, this is not just any infinity. It diverges such that any integral area which includes $t = 0$ has the value of 1. That is (for all $t_- < t_+$),

$$\int_{t_-}^{t_+} dt\, \delta(t) = \begin{cases} 1 & t_- < 0 < t_+ \\ 0 & \text{otherwise} \end{cases} . \tag{5.3}$$

The symbolic plot is shown in Figure 5.3. The ideal impulse, discussed above, can be expressed in terms of a *shifted* Dirac δ-function:

$$F(t) = \Delta(m_o v)\delta(t - t_o). \tag{5.4}$$

The $t - t_o$ argument simply translates the spike of the δ-function so that it occurs at t_o instead of 0.

5.1.2 Point Masses and Point Charges

Physical equations often involve the mass per unit volume $\rho_m(\bar{\mathbf{r}})$ of a region of space. Normally $\rho_m(\bar{\mathbf{r}})$ is a continuous function of position, but with δ-functions, it can also represent point masses. A point mass has a finite amount of mass stuffed inside a single point of space, so the density must be infinite at that point and zero everywhere else.

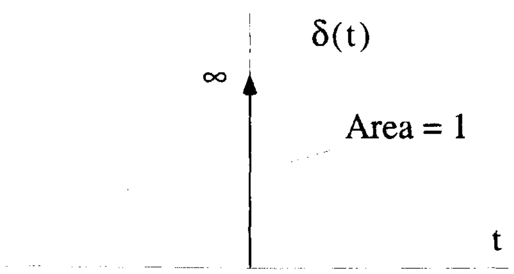


Figure 5.3 The Dirac δ-function, $\delta(t)$

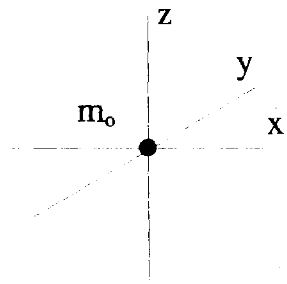


Figure 5.4 A Single Point Mass at the Origin

Integrating the mass density over a volume V gives the total mass enclosed:

$$\int_V d\tau\ \rho_m(\bar{\mathbf{r}}) = \text{ total mass inside } V. \tag{5.5}$$

Thus, if there is a single point of mass m_o located at the origin, as shown in Figure 5.4, any volume integral which includes the origin must give the total mass as m_o. Integrals which exclude the origin must give zero. In mathematical terms:

$$\int_V d\tau\ \rho_m(\bar{\mathbf{r}}) = \begin{cases} m_o & \text{origin included in } V \\ 0 & \text{origin excluded from } V \end{cases}. \tag{5.6}$$

Using Dirac δ-functions, this mass density function becomes

$$\rho_m(\bar{\mathbf{r}}) = m_o\delta(x)\delta(y)\delta(z). \tag{5.7}$$

Equation 5.6 can easily be checked by expanding the integral as

$$\int_V d\tau\ \rho_m(\bar{\mathbf{r}}) = \int dx \int dy \int dz\ m_o\delta(x)\delta(y)\delta(z). \tag{5.8}$$

Application of Equation 5.3 on the three independent integrals gives m_o only when V includes the origin. If, instead of the origin, the point mass is located at the point (x_o, y_o, z_o), shifted arguments are used in each of the δ-functions:

$$\rho_m(\bar{\mathbf{r}}) = m_o\delta(x - x_o)\delta(y - y_o)\delta(z - z_o). \tag{5.9}$$

Dirac δ-functions can also be used, in a similar way, to represent point charges in electromagnetism.

5.2 TWO DEFINITIONS OF $\delta(t)$

There are two common ways to define the Dirac δ-function. The more rigorous approach, from the theory of generalized functions, defines it by its behavior inside integral operations. In fact, the δ-function is actually never supposed to exist outside an integral. In general, scientists and engineers are a bit more lax and use a second definition. They often define the δ-function as the limit of an infinite sequence of

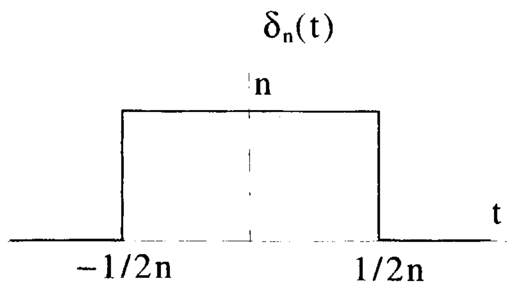


Figure 5.5 The Square Sequence Function

continuous functions. Also, as demonstrated in the two examples of the previous section, they frequently manipulate the δ-function outside of integrals. Usually, there are no problems with this less rigorous approach. However, there are some cases, such as the oscillatory sequence functions described at the end of this section, where the more careful integral approach becomes essential.

5.2.1 $\delta(t)$ as the Limit of a Sequence of Functions

The δ-function can be viewed as the limit of a sequence of functions. In other words,

$$\delta(t) = \lim_{n \to \infty} \delta_n(t), \tag{5.10}$$

where $\delta_n(t)$ is finite for all values of t.

There are many function sequences that approach the Dirac δ-function in this way. The simplest is the square function sequence defined by

$$\delta_n(t) = \begin{cases} n & -1/2n < t < +1/2n \\ 0 & \text{otherwise} \end{cases} \tag{5.11}$$

and shown in Figure 5.5. Clearly, for any value of n

$$\int_{-\infty}^{\infty} dt\ \delta_n(t) = 1, \tag{5.12}$$

and in the limit as $n \to \infty$, $\delta_n(t) = 0$ for all t, except $t = 0$. The first three square sequence functions for $n = 1, 2,$ and 3 are shown in Figure 5.6.

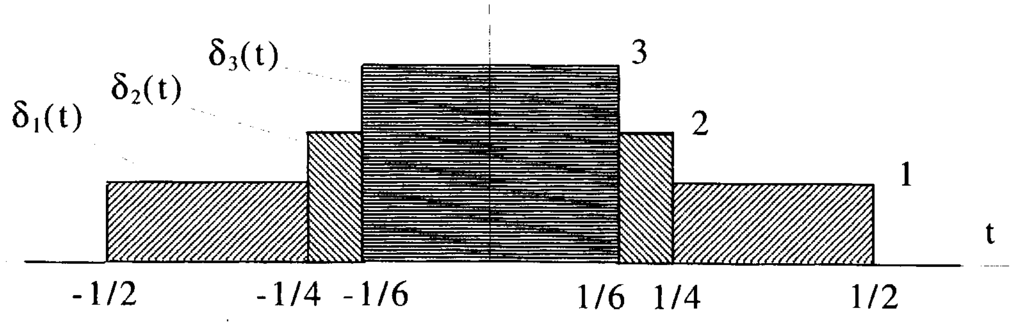


Figure 5.6 The First Three Square Sequence Functions

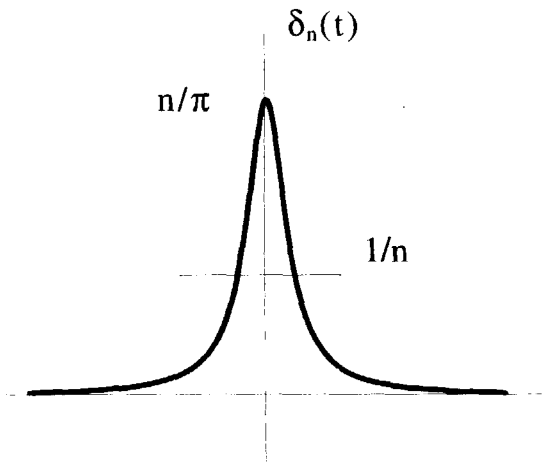


Figure 5.7 The Resonance Sequence Function

There are four other common function sequences that approach the Dirac δ-function. Three of them, the resonance, Gaussian, and sinc squared sequences, are shown in Figures 5.7–5.9 and are mathematically described by

$$\begin{aligned} \text{Resonance:} \quad & \delta_n(t) = \frac{n/\pi}{1 + n^2t^2} \\ \text{Gaussian:} \quad & \delta_n(t) = \frac{n}{\sqrt{\pi}} e^{-n^2t^2} \\ \text{Sinc Squared:} \quad & \delta_n(t) = \frac{\sin^2 nt}{n\pi t^2}. \end{aligned} \tag{5.13}$$

Each of these functions has unit area for any value of n, and it is easy to calculate that in the limit as $n \to \infty$, $\delta_n(t) = 0$ for all $t \neq 0$.

There is one other common sequence, the sinc function sequence, which approaches the δ-function in a completely different manner. The discussion of this requires more rigor, and is deferred until the end of this section.

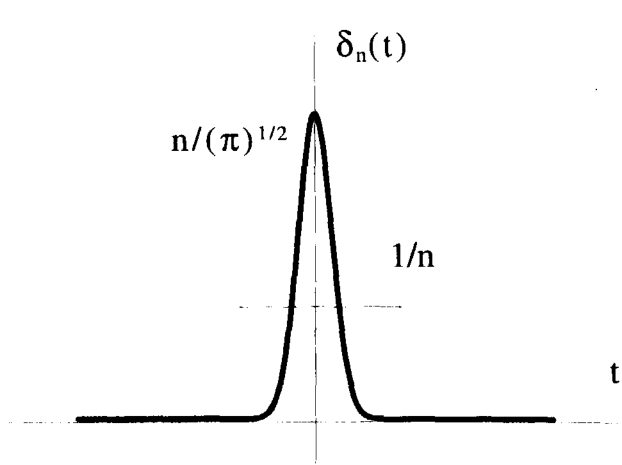


Figure 5.8 The Gaussian Sequence Function

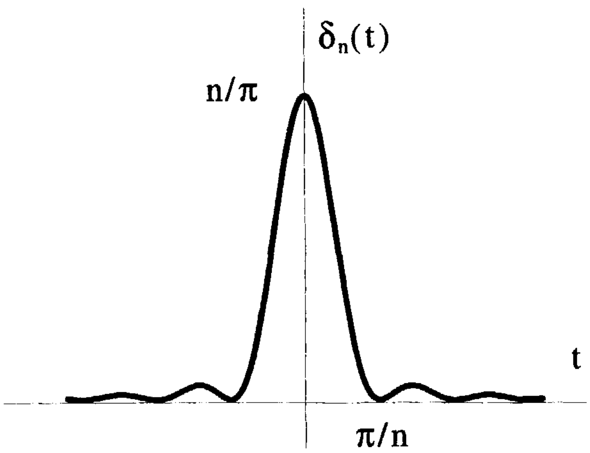


Figure 5.9 The Sinc Squared Sequence Function

5.2.2 Defining $\delta(t)$ by Integral Operations

In mathematics, the δ-function is defined by how it behaves inside an integral. Any function which behaves as $\delta(t)$ in the following equation is *by definition* a δ-function,

$$\int_{t_-}^{t_+} dt\ \delta(t - t_o)f(t) = \begin{cases} f(t_o) & t_- < t_o < t_+ \\ 0 & \text{otherwise} \end{cases}, \tag{5.14}$$

where $t_- < t_+$ and $f(t)$ is any continuous, well-behaved function. This operation is sometimes called a *sifting integral* because it selects the single value $f(t_o)$ out of $f(t)$.

Because this is a definition, it need not be proven, but its consistency with our previous definition and applications of the δ-function must be shown. If $t_- < t_o < t_+$, the range of the integral can be changed to be an infinitesimal region of size 2ϵ centered around t_o, without changing the value of the integral. This is true because $\delta(t - t_o)$ vanishes everywhere except at $t = t_o$. This means

$$\int_{t_-}^{t_+} dt\ \delta(t - t_o)f(t) = \int_{t_o-\epsilon}^{t_o+\epsilon} dt\ \delta(t - t_o)f(t). \tag{5.15}$$

Because $f(t)$ is a continuous function, over the infinitesimal region it is effectively a constant, with the value $f(t_o)$. Therefore,

$$\int_{t_-}^{t_+} dt\ \delta(t - t_o)f(t) = f(t_o)\int_{t_o-\epsilon}^{t_o+\epsilon} dt\ \delta(t - t_o) = f(t_o). \tag{5.16}$$

This "proves" the first part of Equation 5.14. The second part, when t_o is not inside the range $t_- < t < t_+$, easily follows because, in this case, $\delta(t - t_o)$ is zero for the entire range of the integrand.

Figure 5.10 shows a representation of the integration in Equation 5.14. The integrand is a product of a shifted δ-function and the continuous function $f(t)$. Because the δ-function is zero everywhere except at $t = t_o$, the integrand goes to a δ-function located at $t = t_o$, with an area scaled by the value of $f(t_o)$.

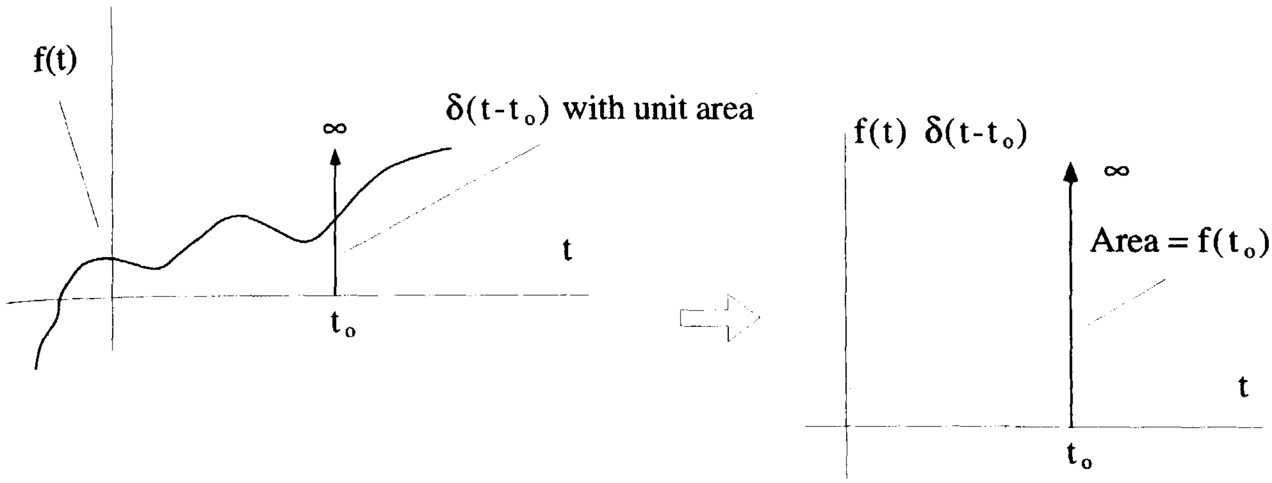


Figure 5.10 A Graphical Interpretation of the Sifting Integral

5.2.3 The Sinc Sequence Function

The sinc function can be used to form another set of sequence functions which, in the limit as $n \longrightarrow \infty$, approach the behavior of a δ-function. The sinc function sequence is shown in Figure 5.11 and is defined by

$$\text{Sinc:} \quad \delta_n(t) = \frac{\sin nt}{\pi t}. \tag{5.17}$$

Notice, however, that as $n \longrightarrow \infty$ this $\delta_n(t)$ does not approach zero for all $t \neq 0$, so the sinc sequence does not have the characteristic infinitely narrow, infinitely tall peak that we have come to expect. How, then, can we claim this sequence approaches a δ-function?

The answer is that the sinc function approaches the δ-function behavior from a completely different route, which can only be understood in the context of the integral definition of the δ-function. In the limit as $n \longrightarrow \infty$, the sinc function oscillates infinitely fast except at $t = 0$. Thus, when the sequence is applied to a continuous function $f(t)$ in a sifting integral, the only contribution which is not canceled out by rapid oscillations comes from the point $t = 0$. The result is, as you will prove in one of the

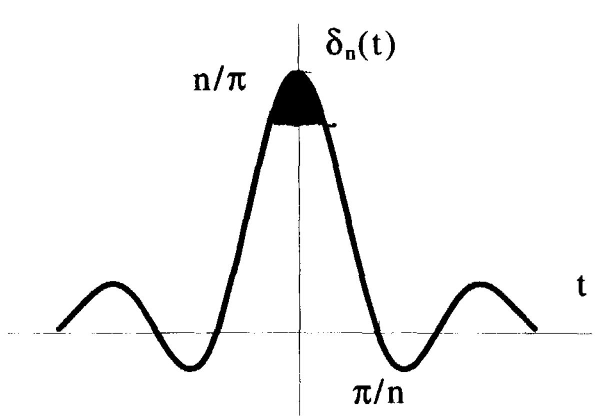


Figure 5.11 The Sinc Sequence Function

exercises at the end of this chapter, the same as you would expect for any δ-function:

$$\int_{t_-}^{t_+} dt \left[\lim_{n\to\infty} \frac{\sin nt}{\pi t}\right] f(t) = \begin{cases} f(0) & t_- < 0 < t_+ \\ 0 & \text{otherwise} \end{cases}. \tag{5.18}$$

This means, in the limit $n \to \infty$, the sinc sequence approaches the δ-function:

$$\lim_{n\to\infty} \frac{\sin nt}{\pi t} = \delta(t). \tag{5.19}$$

5.3 δ-FUNCTIONS WITH COMPLICATED ARGUMENTS

So far, we have only considered the Dirac δ-function with either a single, independent variable as an argument, e.g., $\delta(t)$, or the shifted version $\delta(t - t_o)$. In general, however, the argument could be any function of the independent variable (or variables). It turns out that such a function can always be rewritten as a sum of simpler δ-functions. Three specific examples of how this is accomplished are presented in this section, and then the general case is explored. In each case, the process is the same. The complicated δ-function is inserted into an integral, manipulated, and then rewritten in terms of δ-functions with simpler arguments.

5.3.1 $\delta(-t)$

To determine the properties of $\delta(-t)$, make it operate on any continuous function $f(t)$ inside a sifting integral,

$$\int_{t_-}^{t_+} dt\; \delta(-t) f(t), \tag{5.20}$$

where $t_- < t_+$. The substitution $t' = -t$ transforms this to

$$-\int_{-t_-}^{-t_+} dt'\; \delta(t') f(-t') = \int_{-t_+}^{-t_-} dt'\; \delta(t') f(-t')$$

$$= \begin{cases} f(0) & -t_+ < 0 < -t_- \\ 0 & \text{otherwise} \end{cases}, \tag{5.21}$$

where the last step follows from the integral definition of $\delta(t)$ in Equation 5.14. Therefore, we have

$$\int_{t_-}^{t_+} dt\; \delta(-t) f(t) = \begin{cases} f(0) & t_- < 0 < t_+ \\ 0 & \text{otherwise} \end{cases}. \tag{5.22}$$

But notice this is precisely the result obtained if a $\delta(t)$ were applied to the same function:

$$\int_{t_-}^{t_+} dt\ \delta(t) f(t) = \begin{cases} f(0) & t_- < 0 < t_+ \\ 0 & \text{otherwise} \end{cases}. \tag{5.23}$$

Remember, by definition, anything that behaves like a δ-function inside an integral is a δ-function, so $\delta(-t) = \delta(t)$. This implies the δ-function is an even function.

5.3.2 $\delta(at)$

Now consider another simple variation $\delta(at)$, where a is any positive constant. Again, start with a sifting integral in the form

$$\int_{t_-}^{t_+} dt\ f(t)\delta(at). \tag{5.24}$$

With a variable change of $t' = at$, this becomes

$$\int_{t_-/a}^{t_+/a} dt' \frac{f(t'/a)\delta(t')}{a} = \begin{cases} f(0)/a & t_-/a < 0 < t_+/a \\ 0 & \text{otherwise} \end{cases}, \tag{5.25}$$

so that

$$\int_{t_-}^{t_+} dt\ f(t)\delta(at) = \begin{cases} f(0)/a & t_- < 0 < t_+ \\ 0 & \text{otherwise} \end{cases}. \tag{5.26}$$

Notice again, this is just the definition of the function $\delta(t)$ multiplied by the constant $1/a$, so

$$\delta(ax) = \delta(x)/a \qquad a > 0. \tag{5.27}$$

This derivation was made assuming a positive a. The same manipulations can be performed with a negative a, and combined with the previous result, to obtain the more general expression

$$\delta(ax) = \delta(x)/|a|. \tag{5.28}$$

Notice that our first example, $\delta(t) = \delta(-t)$, can be derived from Equation 5.28 by setting $a = -1$.

5.3.3 $\delta(t^2 - a^2)$

As a bit more complicated argument for the Dirac δ-function, consider $\delta(t^2 - a^2)$. The argument of this function goes to zero when $t = +a$ and $t = -a$, which seems to imply two δ-functions. To test this theory, place the function in the sifting integral

$$\int_{t_-}^{t_+} dt\ f(t)\delta(t^2 - a^2). \tag{5.29}$$

There can be contributions to this integral only at the zeros of the argument of the δ-function. Assuming that the integral range includes both these zeros, i.e., $t_- < -a$ and $t_+ > +a$, this integral becomes

$$\int_{-a-\epsilon}^{-a+\epsilon} dt\ f(t)\delta(t^2 - a^2) + \int_{+a-\epsilon}^{+a+\epsilon} dt\ f(t)\delta(t^2 - a^2), \tag{5.30}$$

which is valid for any value of $0 < \epsilon < a$.

Now $t^2 - a^2 = (t - a)(t + a)$, which near the two zeros can be approximated by

$$t^2 - a^2 = (t - a)(t + a) \approx \begin{cases} (t + a)(-2a) & t \to -a \\ (t - a)(+2a) & t \to +a \end{cases}. \tag{5.31}$$

Formally, these results are obtained by performing a Taylor series expansion of $t^2 - a^2$ about both $t = -a$ and $t = +a$ and keeping terms up to the first order. The Taylor series expansion of $t^2 - a^2$ around an arbitrary point t_o is, to first order, given by

$$t^2 - a^2 \approx t_o^2 - a^2 + \left.\frac{d(t^2 - a^2)}{dt}\right|_{t=t_o} (t - t_o). \tag{5.32}$$

In the limit as $\epsilon \to 0$, Integral 5.30 then becomes

$$\int_{-a-\epsilon}^{-a+\epsilon} dt\ f(t)\delta\big(-2a(t + a)\big) + \int_{+a-\epsilon}^{+a+\epsilon} dt\ f(t)\delta\big(2a(t - a)\big). \tag{5.33}$$

Using the result from the previous section, $\delta(at) = \delta(t)/|a|$, gives

$$\int_{t_-}^{t_+} dt\ f(t)\delta(t^2 - a^2) = \frac{1}{2a}f(-a) + \frac{1}{2a}f(+a). \tag{5.34}$$

Therefore, $\delta(t^2 - a^2)$ is equivalent to the sum of two δ-functions:

$$\delta(t^2 - a^2) = \frac{1}{2a}\delta(t - a) + \frac{1}{2a}\delta(t + a), \tag{5.35}$$

as shown in Figure 5.12.

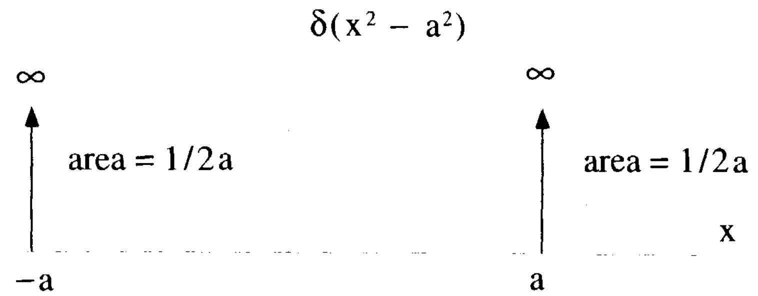


Figure 5.12 The Plot of $\delta(x^2 - a^2)$

5.3.4 The General Case of $\delta(f(t))$

Using the Taylor series approach, the previous example can be easily generalized to an arbitrary argument inside a δ-function:

$$\delta\big(f(t)\big) = \sum_i \left| \frac{1}{df/dt|_{t=t_i}} \right| \delta(t - t_i), \tag{5.36}$$

where the sum over i is a sum over all the zeros of $f(t)$ and t_i is the value of t where each zero occurs.

5.4 INTEGRALS AND DERIVATIVES OF $\delta(t)$

Integrating the δ-function is straightforward because the δ-function is defined by its behavior inside integrals. But, because the δ-function is extremely discontinuous, you might think that it is impossible to talk about its derivatives. While it is true that the derivatives cannot be treated like those of continuous functions, it is possible to talk about them inside integral operations or as limits of the derivatives of a sequence of functions. These manipulations are sometimes referred to as δ-function calculus.

5.4.1 The Heaviside Unit Step Function

Consider the function $H(x)$, which results from integrating the δ-function:

$$H(x) = \int_{-\infty}^{x} dt\ \delta(t). \tag{5.37}$$

$H(x)$ can be interpreted as the area under $\delta(t)$ in the range from $t = -\infty$ to $t = x$, as shown in Figure 5.13. If $x < 0$, the range of integration does not include $t = 0$, and $H(x) = 0$. As soon as x exceeds zero, the integration range includes $t = 0$, and $H(x) = 1$:

$$H(x) = \int_{-\infty}^{x} dt\ \delta(t) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}. \tag{5.38}$$

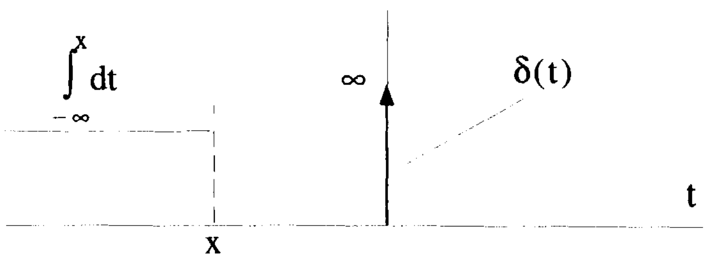


Figure 5.13 The Integration for the Heaviside Step Function

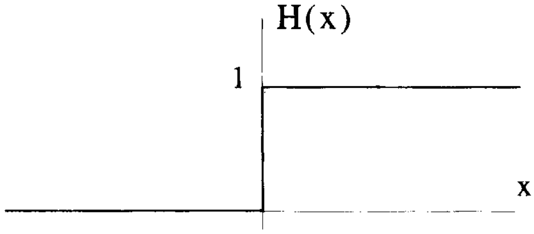


Figure 5.14 The Heaviside Function

Strictly, the value of $H(0)$ is not well defined, but because $\delta(t)$ is an even function, many people define $H(0) = 1/2$. A plot of the Heaviside function is shown in Figure 5.14.

Ironically, the function does not get its name because it is heavy on one side but, instead, from the British mathematician and physicist, Oliver Heaviside.

5.4.2 The Derivative of $\delta(t)$

The derivatives of $\delta(t)$ are, themselves, very interesting functions with sifting properties of their own. To explore these functions, picture the first derivative of $\delta(t)$ as the limit of the derivatives of the Gaussian sequence functions:

$$\frac{d\delta(t)}{dt} = \lim_{n\to\infty} \frac{d\delta_n(t)}{dt} = \lim_{n\to\infty} \frac{d}{dt}\left[\frac{n}{\sqrt{\pi}} e^{-n^2t^2}\right]. \tag{5.39}$$

Figure 5.15 depicts the limiting process.

The first derivative of $\delta(t)$, often written $\delta'(t)$, is called a "doublet" because of its opposing pair of spikes, which are infinitely high, infinitely narrow, and infinitely close together. Like $\delta(t)$, the doublet is rigorously defined by how it operates on other functions inside an integral. The sifting integral of the doublet is

$$\int_{t_-}^{t_+} dt\, f(t)\delta'(t), \tag{5.40}$$

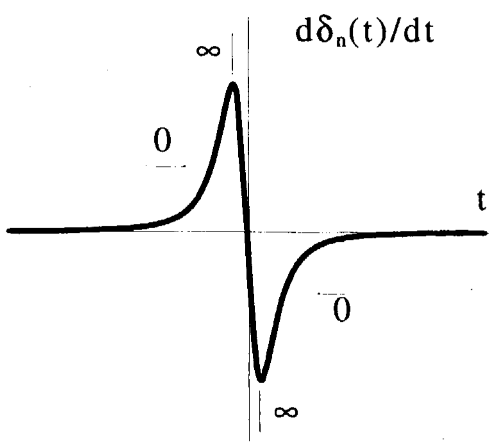


Figure 5.15 The Derivative of $\delta(t)$

where $f(t)$ is any continuous, well-behaved function. This integral can be evaluated using integration by parts. Recall that integration by parts is performed by identifying two functions, $u(t)$ and $v(t)$, and using them in the relation

$$\int_a^b d(u(t))v(t) = u(t)v(t)\Big|_{t=a}^{t=b} - \int_a^b d(v(t))u(t). \tag{5.41}$$

To evaluate Equation 5.40, let $v(t) = f(t)$ and $du = \delta'(t)\,dt$. Then $dv = (df/dt)\,dt$ and $u(t) = \delta(t)$ so that Equation 5.40 becomes

$$\int_{t_-}^{t_+} dt\ f(t)\delta'(t) = f(t)\delta(t)\Big|_{t=t_-}^{t=t_+} - \int_{t_-}^{t_+} dt\ \frac{df(t)}{dt}\delta(t). \tag{5.42}$$

If $t_- \neq 0$ and $t_+ \neq 0$, Equation 5.42 reduces to

$$\int_{t_-}^{t_+} dt\ f(t)\delta'(t) = -\int_{t_-}^{t_+} dt\ \frac{df(t)}{dt}\delta(t), \tag{5.43}$$

because $\delta(t)$ is zero for $t \neq 0$. This is now in the form of the standard sifting integral, so

$$\int_{t_-}^{t_+} dt\ f(t)\delta'(t) = \begin{cases} -f'(0) & t_- < 0 < t_+ \\ 0 & \text{otherwise} \end{cases}. \tag{5.44}$$

The doublet is the sifting function for the negative of the derivative.

This sifting property of the doublet, like the sifting property of $\delta(t)$, has a graphical interpretation. The integrated area under the doublet is zero, so if it is multiplied by a constant, the resulting integral is also zero. However, if the doublet is multiplied by a function that has different values to the right and left of center (i.e., a function with a nonzero derivative at that point), then one side of the integrand will have more area than the other. If the derivative of $f(t)$ is positive, the negative area of the doublet is scaled more than the positive area and the result of the sifting integration is negative. This situation is shown in Figure 5.16. If, on the other hand, the derivative of $f(t)$

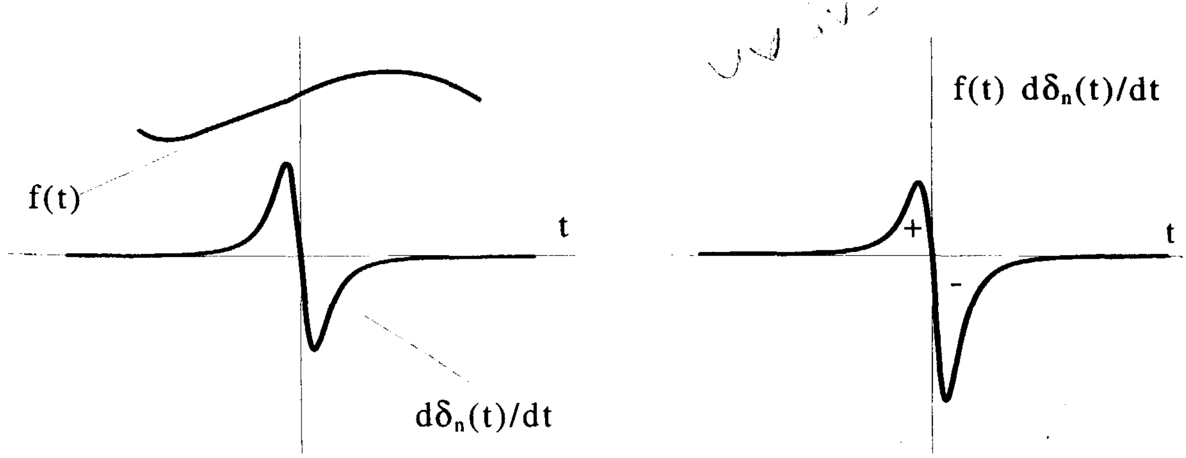


Figure 5.16 The Sifting Property of the Doublet

is negative, the positive area of the doublet is enhanced more than the negative area, and the sifting integration produces a positive result.

Similar arguments can be extended to higher derivatives of the δ-function. The appropriate sifting property is arrived at by applying integration by parts a number of times, until the integral is placed in the form of Equation 5.44.

5.5 SINGULAR DENSITY FUNCTIONS

One of the more common uses of the Dirac δ-function is to describe density functions for singular distributions. Typically, a distribution function describes a continuous, "per unit volume" quantity such as charge density, mass density, or number density. Singular distributions are not continuous, but describe distributions that are confined to sheets, lines, or points. The example of the density of a point mass was briefly described earlier in this chapter.

5.5.1 Point Mass Distributions

The simplest singular mass distribution describes a point of mass m_o located at the origin of a Cartesian coordinate system, as shown in Figure 5.4. The mass density function $\rho_m(x, y, z)$ that describes this distribution must have the units of mass per volume and must be zero everywhere, except at the origin. In addition, any volume integral of the density which includes the origin must give a total mass of m_o, while integrals that exclude the origin must give zero mass. The expression

$$\rho_m(x, y, z) = m_o\delta(x)\delta(y)\delta(z) \tag{5.45}$$

satisfies all of these requirements. Clearly it is zero unless x, y, and z are zero. Therefore integrating ρ_m over a volume that does not include the origin produces zero. Integrating over a volume that contains the origin results in

$$\begin{aligned}\int_V d\tau\, m_o\delta(x)\delta(y)\delta(z) &= \int_{-\epsilon}^{+\epsilon} dx \int_{-\epsilon}^{+\epsilon} dy \int_{-\epsilon}^{+\epsilon} dz\, m_o\delta(x)\delta(y)\delta(z) \\ &= m_o.\end{aligned} \tag{5.46}$$

Because $\int dx\ \delta(x) = 1$, $\delta(x)$ has the inverse dimensions of its argument, or in this case, 1/length. Therefore, $m_o\delta(x)\delta(y)\delta(z)$ has the proper dimensions of mass per unit volume.

To shift the point mass to a different location than the origin, simply use shifted δ-functions:

$$\rho_m = m_o\delta(x - x_o)\delta(y - y_o)\delta(z - z_o). \tag{5.47}$$

This function has the proper dimensions of mass per unit volume, and ρ_m is zero except at the point (x_o, y_o, z_o). The integral over a volume V correctly produces zero mass if V does not include the point (x_o, y_o, z_o) and m_o if it does.

Consider the very same point, but now try to write the mass density in cylindrical coordinates, i.e., $\rho_m(\rho, \phi, z)$. In this system, the coordinates of the mass point are (ρ_o, ϕ_o, z_o) where

$$\begin{aligned} \rho_o &= \sqrt{x_o^2 + y_o^2} \\ \phi_o &= \tan^{-1}(y_o/x_o) \\ z_o &= z_o. \end{aligned} \tag{5.48}$$

A natural guess for $\rho_m(\rho, \phi, z)$ might be

$$m_o\delta(\rho - \rho_o)\delta(\phi - \phi_o)\delta(z - z_o). \tag{5.49}$$

This density clearly goes to zero, unless $\rho = \rho_o$, $\phi = \phi_o$ and $z = z_o$, as it should. However, because ϕ is dimensionless, Equation 5.49 has the dimensions of mass per unit area, which is not correct. Also, the integral over a volume V becomes

$$\int_V d\tau\, \rho_m = \int d\rho \int \rho\, d\phi \int dz\, m_o\delta(\rho - \rho_o)\delta(\phi - \phi_o)\delta(z - z_o). \tag{5.50}$$

This is zero if V does not contain the point (ρ_o, ϕ_o, z_o), but when V does contain the point, the result is $m_o\rho_o$, not m_o as required. This is because the $d\rho$ integration gives

$$\int_{\rho_o - \epsilon}^{\rho_o + \epsilon} d\rho\, \rho\delta(\rho - \rho_o) = \rho_o. \tag{5.51}$$

Therefore, Equation 5.49 is not the proper expression for the mass density in cylindrical coordinates, because it does not have the proper dimensions, and integration does not produce the total mass. From the discussion above, it is clear that the correct density function is

$$\rho_m = \frac{m_o}{\rho}\delta(\rho - \rho_o)\delta(\phi - \phi_o)\delta(z - z_o). \tag{5.52}$$

This example demonstrates an important point. While the point distributions in a Cartesian system can be determined very intuitively, a little more care must be used with non-Cartesian coordinates. In generalized curvilinear coordinates, where $d\tau = h_1h_2h_3dq_1dq_2dq_3$, the expression for a point mass at (q_{1o}, q_{2o}, q_{3o}) is

$$\rho_m(q_1, q_2, q_3) = m_o\frac{\delta(q_1 - q_{1o})}{h_1}\frac{\delta(q_2 - q_{2o})}{h_2}\frac{\delta(q_3 - q_{3o})}{h_3}. \tag{5.53}$$

There is a common shorthand notation used for three-dimensional singular distributions. The three-dimensional Dirac δ-function is defined by

$$\delta^3(\bar{\mathbf{r}} - \bar{\mathbf{r}}_o) \equiv \frac{\delta(q_1 - q_{1o})}{h_1}\frac{\delta(q_2 - q_{2o})}{h_2}\frac{\delta(q_2 - q_{2o})}{h_2}. \tag{5.54}$$

In Cartesian coordinates, this is simply

$$\delta^3(\bar{\mathbf{r}} - \bar{\mathbf{r}}_o) = \delta(x - x_o)\delta(y - y_o)\delta(z - z_o). \tag{5.55}$$

Using this notation, the mass density of point mass located at $\bar{\mathbf{r}}_o$ is, regardless of the coordinate system chosen, given by

$$\rho_m(\bar{\mathbf{r}}) = m_o\delta^3(\bar{\mathbf{r}} - \bar{\mathbf{r}}_o). \tag{5.56}$$

If there are several points with mass m_i at position $\bar{\mathbf{r}}_i$, the density function becomes a sum of δ-functions:

$$\rho_m(\bar{\mathbf{r}}) = \sum_i m_i\delta^3(\bar{\mathbf{r}} - \bar{\mathbf{r}}_i). \tag{5.57}$$

Integrating over a volume V gives

$$\int_V d\tau \sum_i m_i\delta^3(\bar{\mathbf{r}} - \bar{\mathbf{r}}_i), \tag{5.58}$$

which, when evaluated, is simply the sum of all the masses enclosed in V.

5.5.2 Sheet Distributions

Imagine a two-dimensional planar sheet of uniform mass per unit area σ_o, located in the $z = z_o$ plane of a Cartesian system, as shown in Figure 5.17. The intuitive guess for the mass density ρ_m in Cartesian coordinates is

$$\rho_m(x, y, z) = \sigma_o\delta(z - z_o). \tag{5.59}$$

The δ-function makes sure all the mass is in the $z = z_o$ plane. The dimensions are correct, because σ_o has the dimensions of mass per area, and the δ-function adds another 1/length, to give ρ_m the dimensions of mass per unit volume. The real check,

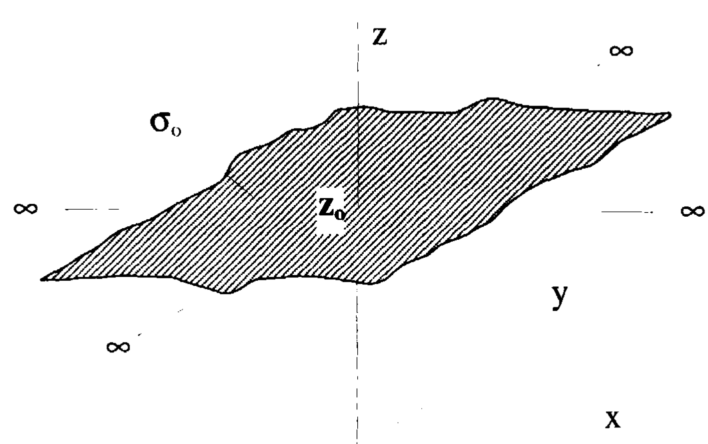


Figure 5.17 Simple Infinite Planar Sheet

however, is the volume integral. In order for Equation 5.59 to be correct, an integral of the surface mass density over some part of the sheet S must give the same total mass as a volume integral of ρ_m, over a volume V, which encloses S:

$$\int_S d\sigma\ \sigma_m(\bar{\mathbf{r}}) = \int_V d\tau\ \rho_m(\bar{\mathbf{r}}). \tag{5.60}$$

For the case described above, Equation 5.60 expands on both sides to

$$\int dx \int dy\ \sigma_o = \int dx \int dy \int dz\ \sigma_o \delta(z - z_o), \tag{5.61}$$

which is indeed true, because the range of the z integration includes the zero of the δ-function. Thus the assumption of Equation 5.59 was correct. Notice how the δ-function effectively converts the volume integral into a surface integral.

Unfortunately, things are not always this easy. The previous example was particularly simple, because the sheet was lying in the plane given by $z = z_o$. Now consider the same sheet positioned in the plane $y = x$, as shown in Figure 5.18. In this case the mass is only located where $y = x$, and the intuitive guess for the mass density is

$$\sigma_o \delta(y - x). \tag{5.62}$$

But our intuition is incorrect in this case. The surface integral on the LHS of Equation 5.60 expands to

$$\int d\sigma\ \sigma_m(\bar{r}) = \int ds \int dz\ \sigma_o, \tag{5.63}$$

where ds is the differential length on the surface in the xy-plane, as shown in Figure 5.19. Notice that

$$\begin{aligned} ds &= \sqrt{(dx)^2 + (dy)^2} \\ &= \sqrt{2}\, dx, \end{aligned} \tag{5.64}$$

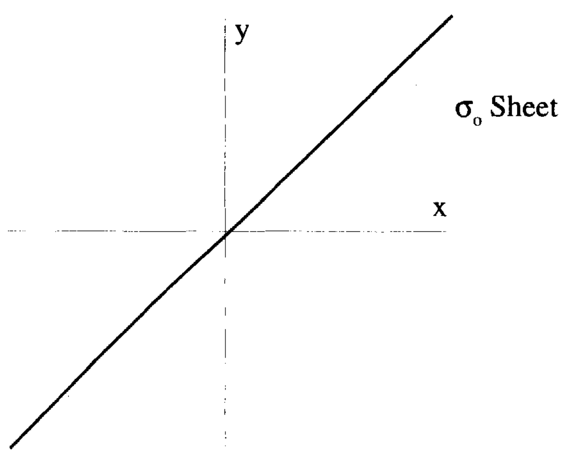


Figure 5.18 A Tilted Planar Mass Sheet

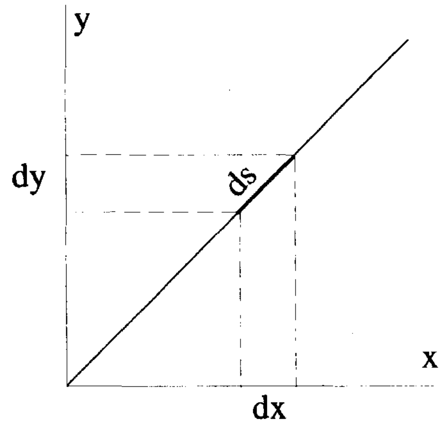


Figure 5.19 The Differential Length ds Along the Surface $x = y$ in the xy-Plane

where the last step follows because $dx = dy$ for this problem. Thus Equation 5.63 becomes

$$\int d\sigma \ \sigma_m(\bar{r}) = \sqrt{2} \int dx \int dz \ \sigma_o. \tag{5.65}$$

If we use the expression in 5.62 for ρ_m, the volume integral on the RHS of Equation 5.60 is

$$\begin{aligned}\int d\tau \ \rho(\bar{r}) &= \int dx \int dy \int dz \ \sigma_o \delta(y - x) \\ &= \int dx \int dz \ \sigma_o,\end{aligned} \tag{5.66}$$

where the last step results from performing the integration over y. Comparing Equations 5.66 and 5.65, we see that they are off by a factor of $\sqrt{2}$!

Where does this discrepancy come from? The problem was our assumption in Equation 5.62 to use a δ-function in the form $\delta(y - x)$. We could have very well chosen $[?]\delta(y - x)$, where [?] is some function that we need to determine. This still makes all the mass lie in the $y = x$ plane. The correct choice of [?] is the one that makes both sides of Equation 5.60 equal. For example, when we make our "guess" for the distribution function for this example, we write

$$\rho_m(x, y, z) = [?]\sigma_o \delta(x - y). \tag{5.67}$$

Then, when we evaluate the RHS of Equation 5.60, we get

$$\begin{aligned}\int d\tau \ \rho(\bar{r}) &= \int dx \int dy \int dz \ \sigma_o [?] \delta(y - x) \\ &= \int dx \int dz \ [?]\sigma_o.\end{aligned} \tag{5.68}$$

In this case, we already showed that the value of [?] is simply the constant $\sqrt{2}$. In more complicated problems, [?] can be a function of the coordinates. This can happen if either the mass per unit area of the sheet is not constant, or if the sheet is not flat. You will see some examples of this in the next section and in the problems at the end of this chapter.

5.5.3 Line Distributions

As a final example of singular density functions, consider the mass per unit volume of a one-dimensional wire of uniform mass per unit length λ_o. The wire is bent to follow the parabola $y = Cx^2$ in the $z = 0$ plane, as shown in Figure 5.20. The factor C is a constant, which has units of 1/length. We will follow the same procedure in constructing the mass density for this wire as we did for the previous example. In this case, the volume integral of the mass density must collapse to a line integral along the wire

$$\int_V d\tau\, \rho_m(\bar{\mathbf{r}}) = \int_L ds\, \lambda_m(s). \tag{5.69}$$

In this equation, s is a variable which indicates parametrically where we are on the wire, and $\lambda_m(s)$ is the mass per unit length of the wire at the position s.

Because all the mass must lie on the wire, we write the mass density function as

$$\rho_m(x, y, z) = [?]\lambda_o\delta(y - Cx^2)\delta(z). \tag{5.70}$$

Here we have used two δ-functions. The $\delta(z)$ term ensures all the mass lies in the $z = 0$ plane, while $\delta(y - Cx^2)$ makes the mass lie along the parabola. As before, we include an unknown factor of [?], which we will have to determine using Equation 5.69.

In terms of Cartesian coordinates, the general expression for the differential arc length ds is

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}. \tag{5.71}$$

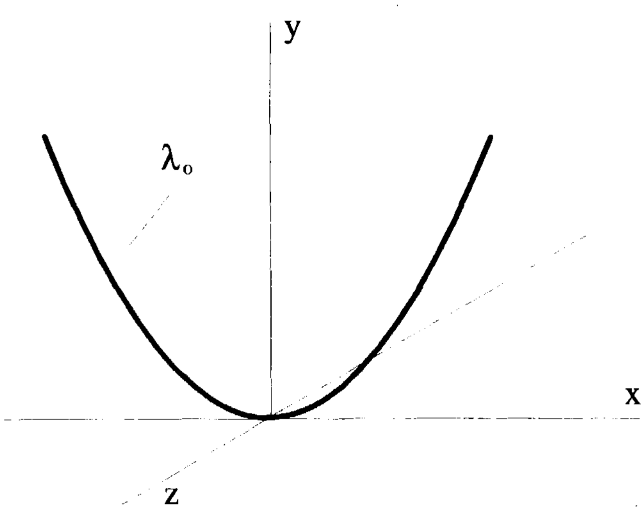


Figure 5.20 Parabolic Line Distribution

Along the wire, $z = 0$ and $y = Cx^2$, so that

$$ds = \sqrt{1 + 4C^2x^2}\, dx. \tag{5.72}$$

Using Equation 5.70 and the fact that $d\tau = dx\, dy\, dz$, Equation 5.69 becomes

$$\int dx \int dy \int dz\, [?]\lambda_o \delta(y - Cx^2)\delta(z) = \int dx\, \sqrt{1 + 4C^2x^2}\, \lambda_o. \tag{5.73}$$

Let's concentrate on the LHS of this equation. The integral over z is easy, because

$$\int dz\, \delta(z) = 1. \tag{5.74}$$

Also, when we do the integral over y, x is held constant, and only one value of y makes the argument of the δ-function vanish, so we have

$$\int dy\, \delta(y - Cx^2) = 1. \tag{5.75}$$

Therefore, Equation 5.73 becomes

$$\int dx\, [?]\lambda_o = \int dx\, \sqrt{1 + 4C^2x^2}\, \lambda_o, \tag{5.76}$$

and the value of [?] clearly must be

$$[?] = \sqrt{1 + 4C^2x^2}. \tag{5.77}$$

Notice in this case that [?] is a function of position. The mass density for the wire is given by

$$\rho_m(x, y, z) = \sqrt{1 + 4C^2x^2}\, \lambda_o \delta(y - Cx^2)\delta(z). \tag{5.78}$$

When we converted the volume integral in Equation 5.73 to a line integral, it was easier to perform the dy integration before the dx integration. This was because when we integrated over y, holding x fixed, the only value of y which made the argument of the δ-function zero was $y = Cx^2$. Another way to look at this is that there is a one-to-one relationship between dx and ds, as shown in Figure 5.21.

If instead, we performed the x integration first, holding the value of y fixed, there are *two* values of x which zero the δ-function argument. In this case, the integral becomes

$$\begin{aligned}\int dx\, \delta(y - Cx^2) &= \frac{1}{C}\int dx\, \delta\left(\left[x - \sqrt{y/C}\right]\left[x + \sqrt{y/C}\right]\right)\\ &= \frac{1}{\sqrt{4Cy}}\int dx\left[\delta\left(x - \sqrt{y/C}\right) + \delta\left(x + \sqrt{y/C}\right)\right].\end{aligned} \tag{5.79}$$

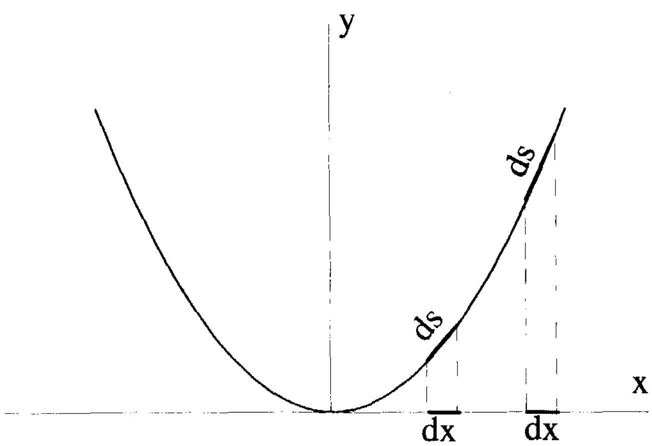


Figure 5.21 The Relation Between dx and ds Along the Parabola

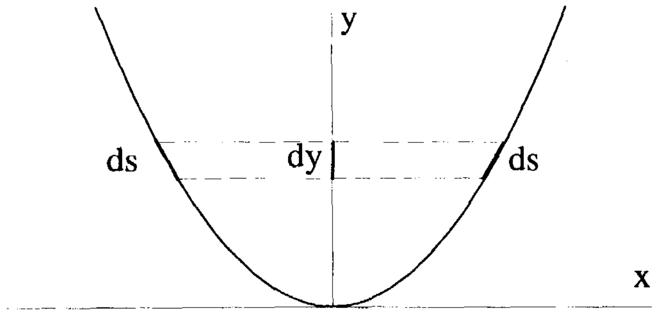


Figure 5.22 The Relation Between dy and ds Along the Parabola

Looking at Figure 5.22 shows what is going on here. There is not a one-to-one relationship between dy and ds. Of course, evaluating the integral in this way produces exactly the same result for $\rho_m(\bar{r})$, as you will prove in one of the exercises of this chapter.

5.6 THE INFINITESIMAL ELECTRIC DIPOLE

The example of the infinitesimal electric dipole is one of the more interesting applications of the Dirac δ-function and makes use of many of its properties.

5.6.1 Moments of a Charge Distribution

In electromagnetism, the distribution of charge density in space $\rho_c(\bar{r})$ can be expanded, in what is generally called a multipole expansion, into a sum of its *moments*. These moments are useful for approximating the potential fields associated with complicated charge distributions in the far field limit (that is, far away from the charges). Each moment is generated by calculating a different volume integral of the charge distribution over all space. Because our goal is not to derive the mathematics of multipole expansions, but rather to demonstrate the use of the Dirac δ-functions, the multipole expansion results are stated here without proof. Derivations can be found in most any intermediate or advanced book on electromagnetism, such as Jackson's *Classical Electrodynamics*.

The lowest term in the expansion is a scalar called the monopole moment. It is just the total charge of the distribution and is determined by calculating the volume integral of ρ_c:

$$Q = \int_{\text{All space}} d\tau \; \rho_c(\bar{\mathbf{r}}). \tag{5.80}$$

The next highest moment is a vector quantity called the dipole moment, which is generated from the volume integral of the charge density times the position vector:

$$\bar{\mathbf{P}} = \int_{\text{All space}} d\tau \; \bar{\mathbf{r}} \rho_c(\bar{\mathbf{r}}). \tag{5.81}$$

The next moment, referred to as the quadrapole moment, is a tensor quantity generated by the integral

$$\bar{\bar{\mathbf{Q}}} = \int_{\text{All space}} d\tau \; \left(3\bar{\mathbf{r}}\,\bar{\mathbf{r}} - |\bar{\mathbf{r}}|^2 \bar{\bar{\mathbf{1}}}\,\right) \rho_c(\bar{\mathbf{r}}). \tag{5.82}$$

In this equation, the quantity $\bar{\mathbf{r}}\,\bar{\mathbf{r}}$ is a dyad, and $\bar{\bar{\mathbf{1}}}$ is the identity tensor. There are an infinite number of higher-order moments beyond these three, but they are used less frequently, usually only in cases where the first three moments are zero.

Far away from the charges, the electric potential can be approximated by summing the contributions from each of the moments. The potential field Φ due to the first few moments is

$$\Phi(\bar{\mathbf{r}}) = \frac{1}{4\pi\epsilon_o} \left[\frac{Q}{r} + \frac{\bar{\mathbf{r}} \cdot \bar{\mathbf{P}}}{r^3} + \frac{\bar{\mathbf{r}} \cdot \bar{\bar{\mathbf{Q}}} \cdot \bar{\mathbf{r}}}{2r^5} + \cdots \right]. \tag{5.83}$$

It is quite useful to know what charge distributions generate just a single term in this expansion, and what potentials and electric fields are associated with them. For example, what charge distribution has just a dipole term (that is, $\bar{\mathbf{P}} \neq 0$) while all other terms are zero ($Q = 0$, $\bar{\bar{\mathbf{Q}}} = 0$, etc.). The Dirac δ-function turns out to be quite useful in describing these particular distributions.

5.6.2 The Electric Monopole

The distribution that generates just the Q/r term in Equation 5.83 is called the electric monopole. As you may have suspected, it is simply the distribution of a point charge at the origin:

$$\rho_{\text{mono}} = q_o \delta^3(\bar{\mathbf{r}}). \tag{5.84}$$

Its monopole moment,

$$Q = \int_{\text{All space}} d\tau\, q_o \delta^3(\overline{\mathbf{r}}) = q_o, \tag{5.85}$$

is simply equal to the total charge. The dipole, quadrapole, and higher moments of this distribution are all zero:

$$\overline{\mathbf{P}} = \int_{\text{All space}} d\tau\, q_o \overline{\mathbf{r}}\, \delta^3(\overline{\mathbf{r}}) = 0 \tag{5.86}$$

$$\overline{\overline{\mathbf{Q}}} = \int_{\text{All space}} d\tau\, \left(3\overline{\mathbf{r}}\,\overline{\mathbf{r}} - r^2\overline{\overline{\mathbf{1}}}\,\right)\delta^3(\overline{\mathbf{r}}) = 0. \tag{5.87}$$

The electric field of a monopole obeys Coulomb's law:

$$\overline{\mathbf{E}} = \frac{1}{4\pi\epsilon_o}\frac{q_o\hat{\mathbf{e}}_r}{r^2}. \tag{5.88}$$

5.6.3 The Electric Dipole

An electric dipole consists of two equal point charges, of opposite sign, separated by some finite distance d_o, as shown in Figure 5.23. The charge density of this system, expressed using Dirac δ-functions, is

$$\rho_{\text{dpl}} = q_o \left[\delta^3\left(\overline{\mathbf{r}} - \frac{d_o}{2}\hat{\mathbf{e}}_x\right) - \delta^3\left(\overline{\mathbf{r}} + \frac{d_o}{2}\hat{\mathbf{e}}_x\right)\right], \tag{5.89}$$

where in this case, the dipole is oriented along the x-axis.

You might be tempted to believe that this distribution has only a dipole moment. Indeed both Q and $\overline{\overline{\mathbf{Q}}}$ are zero, while the dipole moment is given by

$$\begin{aligned}\overline{\mathbf{P}} &= \int d\tau\, q_o\overline{\mathbf{r}}\left[\delta^3\left(\overline{\mathbf{r}} - \frac{d_o}{2}\hat{\mathbf{e}}_x\right) - \delta^3\left(\overline{\mathbf{r}} + \frac{d_o}{2}\hat{\mathbf{e}}_x\right)\right] \\ &= \frac{q_o d_o}{2}\hat{\mathbf{e}}_x + \frac{q_o d_o}{2}\hat{\mathbf{e}}_x \\ &= q_o d_o \hat{\mathbf{e}}_x.\end{aligned} \tag{5.90}$$

The higher-order moments for the dipole distribution, however, do not vanish.

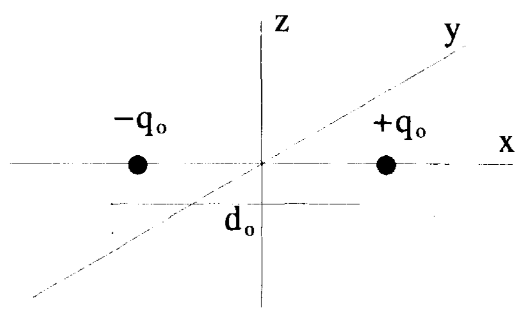


Figure 5.23 The Electric Dipole

The "ideal" dipole refers to a charge distribution that has only a dipole moment, and no other. It is the limiting case of the "physical" dipole, described above. In this limit, the distance between the charges becomes vanishingly small, while the net dipole moment $\bar{\mathbf{P}}$ is held constant. In other words, $d_o \rightarrow 0$ and $q_o \rightarrow \infty$ such that $d_o q_o \equiv p_o$ is held constant. The charge distribution for this situation is

$$\rho_{\text{ideal}} = \lim_{d_o \rightarrow 0} \frac{p_o}{d_o} \left[\delta^3\left(\bar{\mathbf{r}} - \frac{d_o}{2}\hat{\mathbf{e}}_x \right) - \delta^3\left(\bar{\mathbf{r}} + \frac{d_o}{2}\hat{\mathbf{e}}_x \right) \right]. \tag{5.91}$$

Expansion of the δ-functions in terms of Cartesian coordinates gives

$$\rho_{\text{ideal}} = p_o \delta(y)\delta(z) \lim_{d_o \rightarrow 0} \frac{1}{d_o} \left[\delta\left(x - \frac{d_o}{2} \right) - \delta\left(x + \frac{d_o}{2} \right) \right]. \tag{5.92}$$

Notice that this limit is just the definition of a derivative:

$$\lim_{d_o \rightarrow 0} \frac{1}{d_o} \left[\delta\left(x - \frac{d_o}{2} \right) - \delta\left(x + \frac{d_o}{2} \right) \right] = -\frac{d\delta(x)}{dx}. \tag{5.93}$$

This means the charge distribution of an ideal dipole, with a dipole moment of magnitude p_o oriented along the x axis, can be written

$$\rho_{\text{ideal}} = -p_o \frac{d\delta(x)}{dx}\delta(y)\delta(z). \tag{5.94}$$

The electric field from the ideal dipole (and also from a physical dipole in the far field limit) is the solution of

$$\bar{\nabla} \cdot \bar{\mathbf{E}} = \frac{\rho(\bar{\mathbf{r}})}{\epsilon_o} = -\frac{p_o}{\epsilon_o}\frac{d\delta(x)}{dx}\delta(y)\delta(z). \tag{5.95}$$

But we already know the solution for the electric monopole is

$$\bar{\nabla} \cdot \left\{ \frac{1}{4\pi\epsilon_o} \frac{q_o \hat{\mathbf{e}}_r}{r^2} \right\} = \frac{q_o}{\epsilon_o}\delta(x)\delta(y)\delta(z). \tag{5.96}$$

Operating on both sides of Equation 5.96 with $[-d_o \partial/\partial x]$, shows the electric field in Equation 5.95 obeys

$$\bar{\mathbf{E}} = -d_o \frac{\partial}{\partial x} \left\{ \frac{1}{4\pi\epsilon_o} \frac{q_o \hat{\mathbf{e}}_r}{r^2} \right\}. \tag{5.97}$$

Taking the derivative gives the result:

$$\bar{\mathbf{E}} = \frac{p_o}{4\pi\epsilon_o r^3} \left\{ \frac{3x}{r}\hat{\mathbf{e}}_r - \hat{\mathbf{e}}_x \right\}. \tag{5.98}$$

This is the solution for a dipole oriented along the x-axis. In general, the dipole is a vector $\overline{\mathbf{P}}$, which can be oriented in any direction, and Equation 5.98 generalizes to

$$\overline{\mathbf{E}} = \frac{1}{4\pi\epsilon_o r^3}\left\{3(\overline{\mathbf{P}} \cdot \hat{\mathrm{e}}_r)\hat{\mathrm{e}}_r - \overline{\mathbf{P}}\right\}. \tag{5.99}$$

Notice that the magnitude of this field drops off faster with r than the field of the monopole. The higher the order of the moment, the faster its field decays with distance.

5.6.4 Fields from Higher-Order Moments

The technique described above can be used to find the electric field and charge distributions for the ideal electric quadrapole, as well as for all the higher moments. The charge distributions can be constructed using δ-functions, and the fields can be obtained by taking various derivatives of the monopole field. You can practice this technique with the quadrapole moment in an exercise at the end of this chapter.

5.7 RIEMANN INTEGRATION AND THE DIRAC δ-FUNCTION

The δ-function provides a useful, conceptual technique for viewing integration which will become important when we discuss Green's functions in a later chapter. From Equation 5.14, the sifting integral definition of the δ-function, any continuous function $f(y)$ can be written

$$f(y) = \int_{-\infty}^{+\infty} dx\ \delta(x-y)f(x). \tag{5.100}$$

The Riemann definition of integration says that an integral can be viewed as the limit of a discrete sum of rectangles,

$$\int_{-\infty}^{+\infty} dx\ g(x) = \lim_{\Delta x \to 0} \sum_{n=-\infty}^{n=+\infty} g(n\Delta x)\Delta x, \tag{5.101}$$

where Δx is the width of the rectangular blocks which subdivide the area being integrated, and n is an integer which indexes each rectangle. The limiting process increases the number of rectangles, so for well-behaved functions, the approximation becomes more and more accurate as $\Delta x \to 0$. The Riemann definition is pictured graphically in Figures 5.24(a) and (b).

Using the Riemann definition, Equation 5.100 becomes

$$\begin{aligned} f(y) &= \int_{-\infty}^{+\infty} dx\ f(x)\delta(x-y) \\ &= \lim_{\Delta x \to 0} \sum_{n=-\infty}^{n=+\infty} f(n\Delta x)\delta(n\Delta x - y)\Delta x. \end{aligned} \tag{5.102}$$

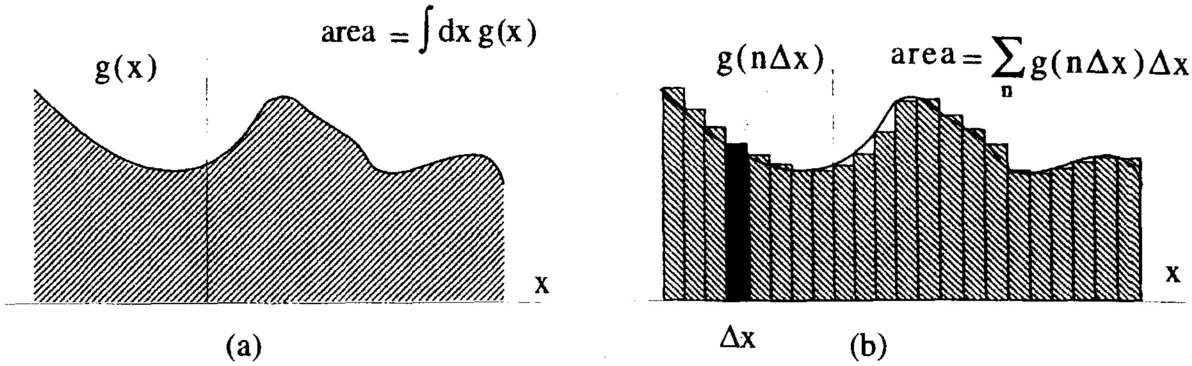


Figure 5.24 Discrete Sum Representation of an Integral

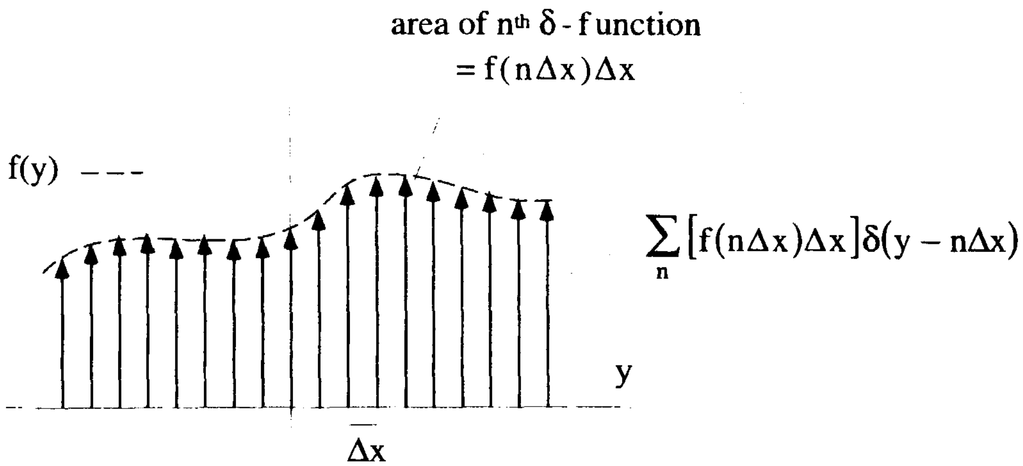


Figure 5.25 The Construction of $f(y)$ from an Infinite Sum of δ-Functions

Equation 5.102 makes a very interesting statement: Any continuous function can be viewed as the sum of an infinite number of δ-functions. Both $f(y)$ and the RHS of Equation 5.102 are functions of y, and are plotted in Figure 5.25 for finite Δx. In this figure, the δ-functions are located at $y = n\Delta x$ and have an area $f(n\Delta x)\Delta x$. The function $f(y)$ is generated by this sum of δ-functions as $\Delta x \to 0$, i.e., as the spacing between the δ-functions and their areas go to zero. Thus an infinite number of infinitesimal area δ-functions, spaced arbitrarily close together, combine to form the continuous function $f(y)$!

EXERCISES FOR CHAPTER 5

1. Simplify the integral

$$\int_{-\infty}^{\infty} dx\, f(x)\delta(-ax + b),$$

where a and b are real, positive constants.

2. Perform these integrations which involve the Dirac δ-function:

i. $\displaystyle\int_{-3}^{0} dx\, \delta(x-1).$

ii. $\displaystyle\int_{-\infty}^{\infty} dx\, (x^2+3)\delta(x-5).$

iii. $\displaystyle\int_{-5}^{5} dx\, x\, \delta(x^2-5).$

iv. $\displaystyle\int_{-5}^{5} dx\, x\, \delta(x^2+5).$

v. $\displaystyle\int_{0}^{2\pi} dx\, \delta(\cos x).$

vi. $\displaystyle\int_{\pi/4}^{3\pi/4} dx\, x^2\, \delta(\cos x).$

vii. $\displaystyle\int_{-\infty}^{\infty} dx\, x^2 \frac{d\delta(x)}{dx}.$

viii. $\displaystyle\int_{-10}^{10} dx\, (x^2+3)\left[\frac{d\delta(x-5)}{dx}\right].$

3. Determine the integral properties of the "triplet" $d^2\delta(t)/dt^2$ by evaluating the integrals

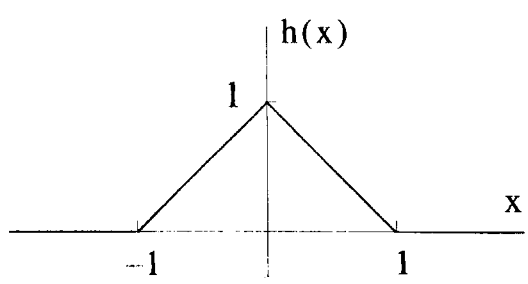
i. $\displaystyle\int_{-\infty}^{\infty} dt \left(\frac{d^2\delta(t)}{dt^2}\right).$

ii. $\displaystyle\int_{-\infty}^{\infty} dt\, f(t) \left(\frac{d^2\delta(t-t_o)}{dt^2}\right).$

4. The function $h(x)$ is generated from the function $g(x)$ by the integral

$$h(x) = \int_{-\infty}^{\infty} dx'\, g(x') \frac{d\delta(x'-x)}{dx'}.$$

If $h(x)$ is the triangular pulse shown below, find and plot $g(x)$.



5. Given that

$$f(x) = \begin{cases} 0 & x < 0 \\ e^x - 1 & x > 0 \end{cases},$$

find and plot the first and second derivatives of $f(x)$.

6. An ideal impulse moving in space and time can be described by the function,

$$f(x,t) = I_o \delta(x - v_o t),$$

where v_o is a constant. Make a three-dimensional plot for $f(x,t)$ vs. x and t to show how this impulse propagates. What are the dimensions of v_o? How does the plot change if

$$f(x,t) = I_o \delta(x - a_o t^2/2),$$

where a_o is a constant? What are the dimensions of a_o?

7. Consider the sinc function sequence:

$$\delta_n(t) = \sin(n\pi)/(\pi x).$$

(a) On the same graph, plot three of these functions with $n = 1$, $n = 10$, and $n = 100$.

(b) Prove that the limit of the sinc sequence functions acts like a δ-function by deriving the relation:

$$\int_{t_-}^{t_+} dt \left[\lim_{n \to \infty} \frac{\sin nt}{\pi t} \right] f(t) = \begin{cases} f(0) & t_- < 0 < t_+ \\ 0 & \text{otherwise} \end{cases}. \qquad (5.103)$$

As a first step, try making the substitution $y = nt$. You will need to use the identity:

$$\int_{-\infty}^{\infty} dx \frac{\sin x}{x} = \pi.$$

8. The function $\delta(\cos x)$ can be written as a sum of Dirac δ-functions

$$\delta(\cos x) = \sum_n a_n \delta(x - x_n).$$

Find the range for n and the values for the a_n and the x_n.

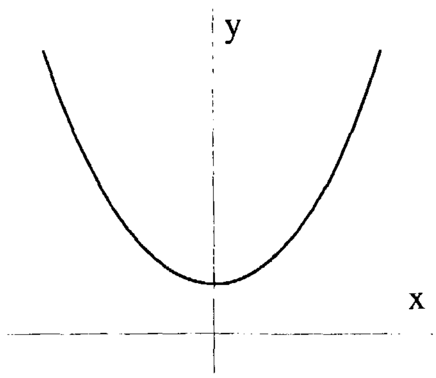
9. A single point charge q_o is located at $(1, 1, 0)$ in a Cartesian coordinate system, so that its charge density can be expressed as

$$\rho_c(x, y, z) = q_o\, \delta(x - 1)\, \delta(y - 1)\, \delta(z).$$

(a) What is $\rho_c(\rho, \theta, z)$, its charge density in cylindrical coordinates?

(b) What is $\rho_c(r, \theta, \phi)$, its charge density in spherical coordinates?

10. An infinitely long, one-dimensional wire of mass per unit length λ_o is bent to follow the curve $y = A\cosh(Bx)$, as shown below.



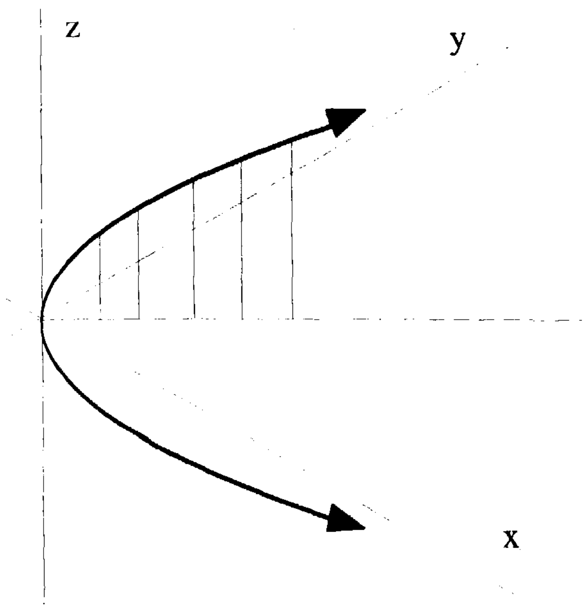
Find an expression for the mass per unit volume $\rho(x, y, z)$. Express your answer two ways:

(a) As the product of two δ-functions.

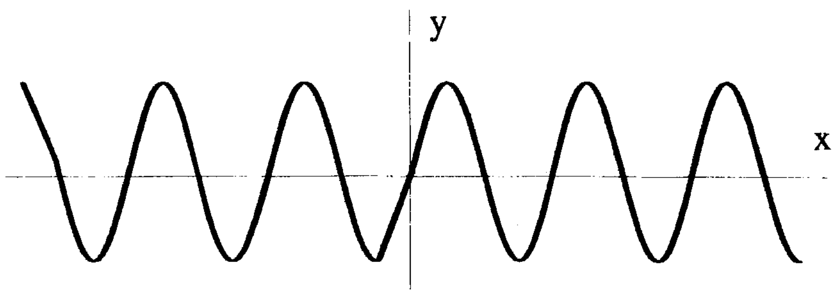
(b) As a sum of two terms, each the product of two δ-functions.

Be sure to check the dimensions of your answers.

11. An infinitely long, one-dimensional wire of mass per unit length λ_o is bent to follow the line formed by the intersection of the surface $x = y$ with the surface $y = z^2$, as shown in the figure below. Find an expression for $\rho_m(x, y, z)$, the mass per unit volume of the wire.



12. An infinitely long, one-dimensional wire with a constant mass per unit length λ_o is bent to follow the curve $y = \sin x$ in the $z = 0$ plane.



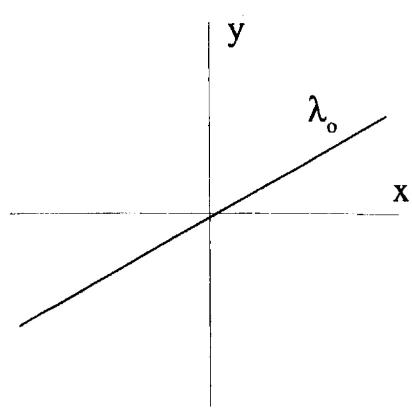
Determine the mass density $\rho_m(x, y, z)$ that describes this mass distribution.

13. A wire of mass per unit length λ_o is bent to follow the shape of a closed ellipse that lies in the xy-plane and is given by the expression

$$x^2 + 2y^2 = 4.$$

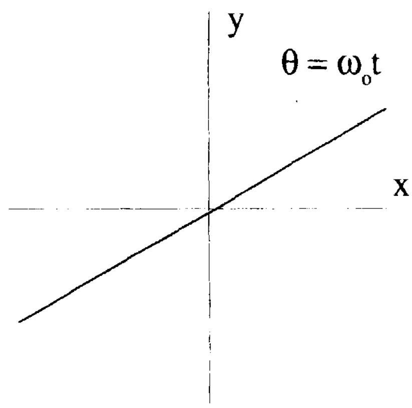
Express $\rho_m(x, y, z)$, the mass per unit volume of this object, using Dirac δ-functions. Show that your expression has the proper dimensions. There is more than one way to express the answer to this problem. Identify the most compact form.

14. An infinite, one-dimensional bar of mass per unit length λ_o lies along the line $y = m_o x$ in the $z = 0$ plane.



(a) Determine the mass per unit volume $\rho(x, y, z)$ of this bar.

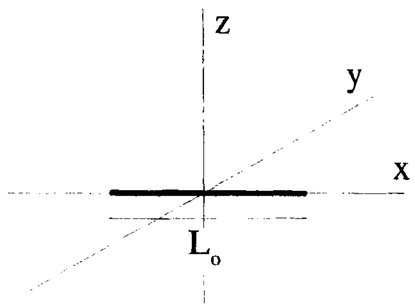
(b) Now consider the situation where the bar is rotating about the z-axis at a constant angular velocity ω_o so that the angle the bar makes with respect to the x-axis is given by $\theta = \omega_o t$, as shown below. Find an expression for the time-dependent mass density $\rho(x, y, z, t)$.



15. A charge Q_o is evenly distributed along the x-axis from $x = -L_o/2$ to $x = L_o/2$, as shown below.

 (a) Using the Heaviside step function, what is the charge density $\rho_c(x, y, z)$, expressed using Cartesian coordinates?

 (b) What is this charge density in cylindrical coordinates?



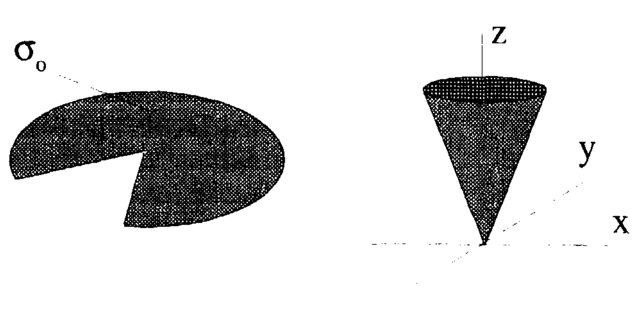
16. Using δ-functions and the Heaviside step function, express the charge density $\rho_c(\bar{\mathbf{r}})$ of a uniformly charged cylindrical shell of radius r_o and length L_o. The total charge on the surface of the shell is Q_o.

17. An infinite, two-dimensional sheet with mass per unit area σ_o is bent to follow the surface $y = x^3$.

 (a) Make a plot of the curve made by the intersection of this sheet and the plane given by $z = 0$.

 (b) Determine the mass per unit volume $\rho_m(x, y, z)$.

18. Express the mass density $\rho_m(\rho, \theta, z)$ of a conical surface that is formed by cutting a pie-shaped piece from an infinite, uniform two-dimensional sheet of mass per unit area σ_o and joining the cut edges. The conical surface that results lies on the surface $\rho = a_o\, z$ where (ρ, θ, z) are the standard cylindrical coordinates.



19. An infinite, two-dimensional sheet with mass per unit area σ_o is bent to follow the surface $xy = 1$ in a Cartesian coordinate system.

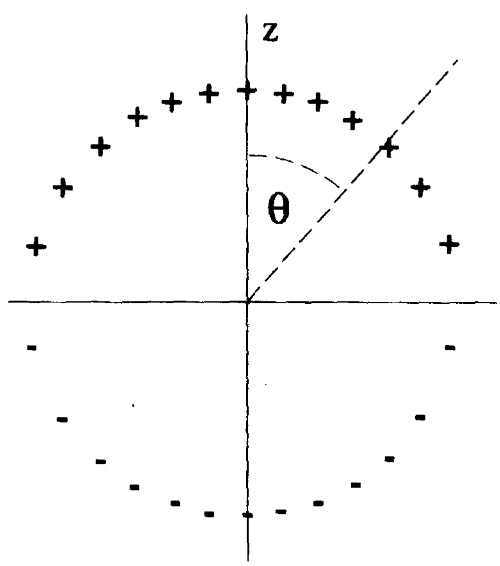
(a) Using the hyperbolic coordinates developed in Exercise 13 of Chapter 3, express the mass density $\rho_m(u, v, z)$ for this sheet.

(b) Using the equations relating the coordinates, convert your answer to part (a) above to Cartesian coordinates.

(c) Now, working from scratch in a Cartesian system, obtain $\rho_m(x, y, z)$ by requiring that this density function take the volume integral over all space to a surface integral over the hyperbolic surface.

20. Express the mass density $\rho_m(\bar{\mathbf{r}})$ for a spherical sheet of radius r_o, with constant mass per unit area σ_o.

21. A dipole electric field is generated outside the surface of a sphere, if the charge per unit area on the surface of that sphere is distributed proportionally to $\cos(\theta)$. If the sphere has a radius r_o and there is a total charge of $+Q_o$ on the upper hemisphere and $-Q_o$ on the lower hemisphere, what is the expression for the charge density $\rho_c(r, \theta, \phi)$ in spherical coordinates?



22. In a two-dimensional Cartesian coordinate system, the mass density $\rho_m(\bar{\mathbf{r}})$ of a pair of point masses is given by

$$\rho_m(x_1, x_2) = m_o\delta(x_1 + 1)\delta(x_2) + m_o\delta(x_1 - 1)\delta(x_2).$$

Evaluate the integrals:

i. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1 dx_2\, \rho_m(x_1, x_2).$

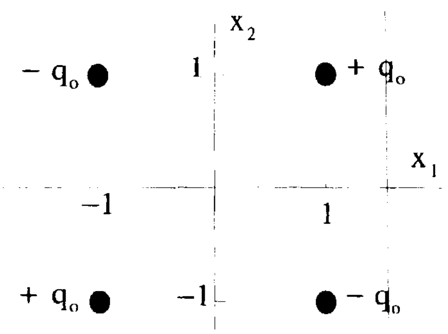
ii. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1 dx_2\, \bar{\mathbf{r}}\, \rho_m(x_1, x_2).$

iii. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_1 dx_2 \, [\bar{\mathbf{r}} \cdot \bar{\mathbf{r}}] \, \rho_m(x_1, x_2).$

iv. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_1 dx_2 \, [\bar{\mathbf{r}} \, \bar{\mathbf{r}}] \, \rho_m(x_1, x_2).$

23. Prove that the monopole and quadrapole moments of any dipole ("physical" or "ideal") are zero.

24. A quadrapole charge distribution consists of four point charges in the x_1x_2-plane as shown below.



(a) Using Dirac δ-functions express the charge density, $\rho_c(x_1, x_2, x_3)$, of this distribution.

(b) The quadrapole moment of this charge distribution is a second rank tensor given by

$$\overline{\overline{\mathbf{Q}}} = Q_{ij} \, \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j.$$

The elements of the quadrapole tensor are given by the general expression

$$Q_{ij} = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3 \, \rho_c(x_1, x_2, x_3) \left[3x_i x_j - (x_k x_k)\delta_{ij}\right],$$

where δ_{ij} is the Kronecker delta. In particular,

$$Q_{22} = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3 \, \rho_c(x_1, x_2, x_3) \left[3x_2 x_2 - (x_1^2 + x_2^2 + x_3^2)\right].$$

Evaluate all the elements of the quadrapole tensor for the charge distribution shown in the figure above.

(c) Does this charge distribution have a dipole moment?

(d) Find the coordinate system in which this quadrapole tensor is diagonal. Express the elements of $\overline{\overline{\mathbf{Q}}}$ in this system.

25. An ideal quadrapole has a charge density $\rho_c(x, y, y)$ that is zero everywhere except at the origin. It has zero total charge, zero dipole moment, and a nonzero quadrapole moment.

(a) Show that if $\rho_c(x, y, z)$ has the form

$$\rho_c(x, y, z) = [?]\frac{d\delta(x)}{dx}\frac{d\delta(y)}{dy}\delta(z),$$

it satisfies the above requirements for an ideal quadrapole. Evaluate [?] so that the elements of the quadrapole moment tensor for this distribution are the same as the quadrapole elements in Exercise 24.

(b) Determine the electric field produced by this ideal quadrapole.