Section 3.10 of "Mathematical Methods in the Physical Sciences" by M.L. Boas, 2nd Edition (John Wiley & Sons, N.Y., 1983)

9. Find the largest triangle that can be inscribed in the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ (assume the triangle symmetric about one axis of the ellipse with one side perpendicular to this axis).

10. Complete Example 4 above.

11. Find the shortest distance from the origin to the line of intersection of the planes $2x + y - z = 1$ and $x - y + z = 2$.

12. Find the right triangular prism of given volume and least area if the base is required to be a right triangle.

10. ENDPOINT OR BOUNDARY POINT PROBLEMS

So far we have been assuming that if there is a maximum or minimum point, calculus will find it. Some simple examples (see Figures 10.1 to 10.4) show that this may not be true. Suppose, in a given problem, x can have values only between 0 and 1; this sort of restriction occurs frequently in applications. For example, if $x = |\cos\theta|$, the *graph* of $f(x) = 2 - x^2$ exists for all real x, but it has no meaning if $x = |\cos\theta|$, except for x between 0 and 1. As another example, suppose x is the length of a rectangle whose perimeter is 2; then $x < 0$ is meaningless in this problem since x is a length, and $x > 1$ is impossible because the perimeter is 2. Let us ask for the largest and smallest values of each of the functions in Figures 10.1 to 10.4 for $0 \le x \le 1$. In Figure 10.1, calculus will give us the minimum point, but the maximum of $f(x)$ *for x between* 0 *and* 1 occurs at $x = 1$ and cannot be obtained by calculus, since $f'(x) \neq 0$ there. In Figure 10.2, both the maximum and the minimum of $f(x)$ are at endpoints, the maximum at $x = 0$ and the minimum at $x = 1$. In Figure 10.3 a relative maximum at P and a relative minimum at Q are given by calculus, but the absolute minimum between 0 and 1 occurs at $x = 0$, and the absolute maximum at $x = 1$. Here is a practical example of this sort of

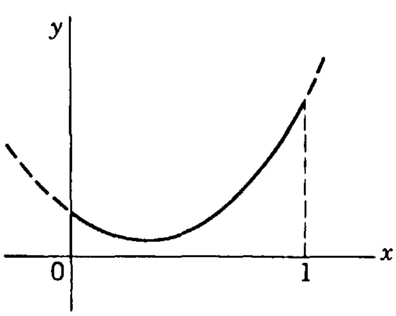


FIGURE 10.1

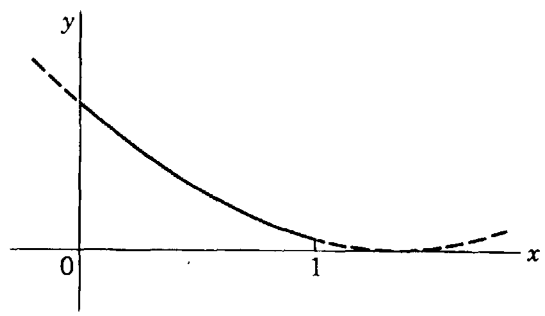


FIGURE 10.2

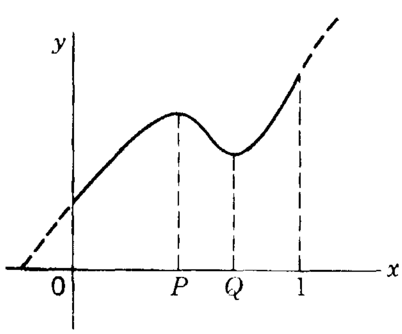


FIGURE 10.3

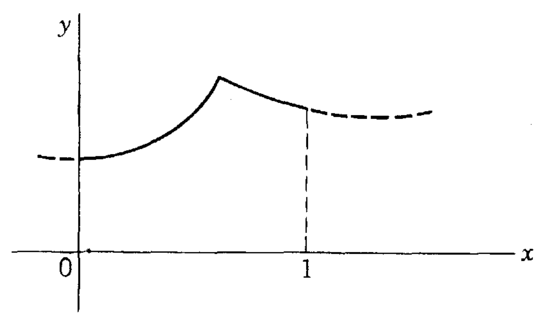


FIGURE 10.4

function. It is said that geographers used to give as the highest point in Florida the top of the highest hill; then it was found that the highest point is on the Alabama border! [See H. A. Thurston, *American Mathematical Monthly*, vol. 68 (1961), pp. 650–652.] Figure 10.4 illustrates another way in which calculus may fail to give us a desired maximum or minimum point; here the derivative is discontinuous at the maximum point.

These are difficulties we must watch out for whenever there is any restriction on the values any of the variables may take (or any discontinuity in the functions or their derivatives). These restrictions are not usually stated in so many words; you have to see them for yourself. For example, if $x^2 + y^2 = 25$, x and y are both between -5 and $+5$. If $y^2 = x^2 - 1$, then $|x|$ must be greater than or equal to 1. If $x = \csc\theta$, where θ is a first-quadrant angle, then $x \geq 1$. If $y = \sqrt{x}$, y' is discontinuous at the origin.

Example 1. A piece of wire 40 cm long is to be used to form the perimeters of a square and a circle in such a way as to make the total area (of square and circle) a maximum. Call the radius of the circle r; then the circumference of the circle is $2\pi r$. A length $40 - 2\pi r$ is left for the four sides of the square, so one side is $10 - \frac{1}{2}\pi r$. The total area is

$$A = \pi r^2 + (10 - \tfrac{1}{2}\pi r)^2.$$

Then

$$\frac{dA}{dr} = 2\pi r + 2(10 - \tfrac{1}{2}\pi r)(-\tfrac{1}{2}\pi) = 2\pi r\left(1 + \frac{\pi}{4}\right) - 10\pi.$$

If $dA/dr = 0$, we get

$$r\left(1 + \frac{\pi}{4}\right) = 5, \qquad r = 2.8, \qquad A = 56+.$$

Now we might think that this is the maximum area. But let us apply the second derivative test to see whether we have a maximum. We find

$$\frac{d^2A}{dr^2} = 2\pi\left(1 + \frac{\pi}{4}\right) > 0;$$

we have found the *minimum* area! The problem asks for a maximum. One way to find it would be to sketch A as a function of r and look at the graph to see where A has its largest value. A simpler way is this. A is a continuous function of r with a continuous derivative. If there were an interior maximum (that is, one between $r = 0$ and $2\pi r = 40$), calculus would find it. Therefore the maximum must be at one end or the other.

$$\text{At} \quad r = 0, \qquad A = 100.$$
$$\text{At} \quad 2\pi r = 40, \qquad r = 6.37, \qquad A = \pi(6.37)^2 = 127+.$$

We see that A takes its largest value at $r = 6.37$; $A = 127+$ is then the desired maximum. It corresponds to using all the wire to make a circle; the side of the square is zero.

A similar difficulty can arise in problems with more variables.

Example 2. The temperature in a rectangular plate bounded by $x = 0$, $y = 0$, $x = 3$, and $y = 5$ is

$$T = xy^2 - x^2y + 100.$$

Find the hottest and coldest points of the plate.

We first set the partial derivatives of T equal to zero to find any interior maxima and minima. We get

$$\frac{\partial T}{\partial x} = y^2 - 2xy = 0,$$

$$\frac{\partial T}{\partial y} = 2xy - x^2 = 0.$$

The only solution of these equations is $x = y = 0$, for which $T = 100$.

We must next ask whether there are points around the boundary of the plate where T has a value larger or smaller than 100. To see that this might happen, think of a graph of T plotted as a function of x and y; this is a surface above the (x, y) plane. The mathematical surface does not have to stop at $x = 3$ and $y = 5$, but it has no meaning for our problem beyond these values. Just as for the curves in Figures 10.1 to 10.4, the graph of the temperature may be increasing or decreasing as we cross a boundary; calculus will not then give us a zero derivative even though the temperature at the boundary may be larger (or smaller) than at other points of the plate. Thus we must consider the complete boundary of the plate (*not* just the corners!). The lines $x = 0$, $y = 0$, $x = 3$, and $y = 5$ are the boundaries; we consider each of them in turn. On $x = 0$ and $y = 0$ the temperature is 100. On the line $x = 3$, we have

$$T = 3y^2 - 9y + 100.$$

We can use calculus to see whether T has maxima or minima as a function of y along this line. We have

$$\frac{dT}{dy} = 6y - 9 = 0,$$

$$y = \tfrac{3}{2}, \qquad T = 93\tfrac{1}{4}.$$

Similarly, along the line $y = 5$, we find

$$T = 25x - 5x^2 + 100,$$

$$\frac{dT}{dx} = 25 - 10x = 0,$$

$$x = \tfrac{5}{2}, \qquad T = 131\tfrac{1}{4}.$$

Finally, we must find T at the corners.

$$\text{At} \quad (0, 0), (0, 5), \text{and } (3, 0), \qquad T = 100.$$

$$\text{At} \quad (3, 5), \qquad T = 130.$$

Putting all our results together, we see that the hottest point is $(\frac{5}{2}, 5)$ with $T = 131\frac{1}{4}$, and the coldest point is $(3, \frac{3}{2})$ with $T = 93\frac{1}{4}$.

Example 3. Find the point or points closest to the origin on the surfaces

$$(10.1)\qquad \begin{aligned} &\text{(a)}\quad x^2 - 4yz = 8,\\ &\text{(b)}\quad z^2 - x^2 = 1. \end{aligned}$$

We want to minimize $f = x^2 + y^2 + z^2$ subject to a condition [(a) or (b)]. If we eliminate x^2 in each case, we have

$$(10.2)\qquad \begin{aligned} &\text{(a)}\quad f = 8 + 4yz + y^2 + z^2,\\ &\text{(b)}\quad f = z^2 - 1 + y^2 + z^2 = 2z^2 + y^2 - 1. \end{aligned}$$

In both (a) and (b) the *mathematical function* $f(y, z)$ is defined for all y and z. *For our problems*, however, this is not true. In (a), since $x^2 \geq 0$, we have $x^2 = 8 + 4yz \geq 0$ so we are interested in minimum values of $f(y, z)$ in (a) only in the region $yz \geq -2$. [Compare Example 2 where $T(x, y)$ was of interest only inside a rectangle.] Thus we look for "interior" minima in (a) satisfying $yz \geq -2$; then we substitute $z = -2/y$ into (10.2a) and find any minima on the boundary of the region of interest. In (b), since $x^2 = z^2 - 1 \geq 0$, we must have $z^2 \geq 1$. Again we try to find "interior" minima satisfying $z^2 \geq 1$; then we set $z^2 = 1$ and look for boundary minima. We now carry out these steps.

From (10.2a), we find

$$(10.3\text{a})\qquad \left.\begin{aligned} \frac{\partial f}{\partial y} &= z + 2y = 0,\\ \frac{\partial f}{\partial z} &= y + 2z = 0, \end{aligned}\right\} \qquad y = z = 0.$$

These values satisfy the condition $yz > -2$ and so give points inside the region of interest. We find from (10.1a), $x^2 = 8$, $x = \pm 2\sqrt{2}$; the points are $(\pm 2\sqrt{2}, 0, 0)$ at distance $2\sqrt{2}$ from the origin. Next we consider the boundary $x = 0$, $z = -2/y$; from (10.2a),

$$f = 0 + y^2 + \frac{4}{y^2}, \qquad \frac{df}{dy} = 2y - \frac{8}{y^3} = 0,$$

$$y^4 = 4, \qquad y = \pm\sqrt{2}, \qquad z = -2/y = \mp\sqrt{2}.$$

Remembering that $x = 0$, we have the points $(0, \sqrt{2}, -\sqrt{2})$ and $(0, -\sqrt{2}, \sqrt{2})$ at distance 2 from the origin. Since $2 < 2\sqrt{2}$, these boundary points are closest to the origin.

(10.4a) Answer to (a): $(0, \sqrt{2}, -\sqrt{2})$, $(0, -\sqrt{2}, \sqrt{2})$.

From (10.2b) we find

$$(10.3\text{b})\qquad \left.\begin{aligned} \frac{\partial f}{\partial y} &= 2y = 0,\\ \frac{\partial f}{\partial z} &= 4z = 0, \end{aligned}\right\} \qquad y = z = 0.$$

Since $z = 0$ does not satisfy $z^2 \geq 1$, there is no minimum point *inside the region of interest*, so we look at the boundary $z^2 = 1$. From (10.1b), $x = 0$, and from (10.2b)

$$f = y^2 + 1, \qquad \frac{df}{dy} = 2y = 0, \qquad y = 0.$$

Thus we find the points $(0, 0, \pm 1)$ at distance 1 from the origin. Since the geometry tells us that there must be a point or points closest to the origin, and calculus tells us that these are the only possible minimum points, these must be the desired points.

(10.4b) Answer to (b): $(0, 0, \pm 1)$.

In both these problems, we could have avoided having to consider the boundary of the region of interest by eliminating z to obtain f as a function of x and y. Since x and y are allowed by (10.1a) or (10.1b) to take *any* values, there are no boundaries to the region of interest. In (b) this is a satisfactory method; in (a) the algebra is complicated. In both problems, Lagrange multipliers offer a more routine method. For example, in (a) we write

$$F = x^2 + y^2 + z^2 + \lambda(x^2 - 4yz);$$

$$\frac{\partial F}{\partial x} = 2x(1 + \lambda) = 0, \qquad x = 0 \text{ or } \lambda = -1;$$

$$\frac{\partial F}{\partial y} = 2y - 4\lambda z = 0; \qquad \text{if } \lambda = -1, y = z = 0, x^2 = 8;$$

$$\frac{\partial F}{\partial z} = 2z - 4\lambda y = 0; \qquad \text{if } x = 0, \lambda = \frac{y}{2z} = \frac{z}{2y}, y^2 = z^2 = 2.$$

We obtain the same results as above, namely, the points $(\pm 2\sqrt{2}, 0, 0)$, $(0, \pm\sqrt{2}, \mp\sqrt{2})$; the points $(0, \sqrt{2}, -\sqrt{2})$, $(0, -\sqrt{2}, \sqrt{2})$ are closer to the origin by inspection. Part (b) can be done similarly (Problem 15).

We see that using Lagrange multipliers may simplify maximum and minimum problems. However, the Lagrange multiplier method still relies on calculus; consequently, it can work only if the maximum and minimum can be found by calculus using *some* set of variables (x and y, *not* y and z, in Example 3). For example, a problem in which the maximum or minimum occurs at endpoints in all variables cannot be done by *any* method that depends on setting derivatives equal to zero.

Example 4. Find the maximum value of $y - x$ for nonnegative x and y if $x^2 + y^2 = 1$.

Here we must have both x and y between 0 and 1. Then the values $y = 1$ and $x = 0$ give $y - x$ its largest value; these are both endpoint values which cannot be found by calculus.

PROBLEMS, SECTION 10

1. Find the shortest distance from the origin to $x^2 - y^2 = 1$.

2. Find the largest and smallest distances from the origin to the conic whose equation is $5x^2 - 6xy + 5y^2 - 32 = 0$ and hence determine the lengths of the semiaxes of this conic.

3. Repeat Problem 2 for the conic $6x^2 + 4xy + 3y^2 = 28$.

Find the shortest distance from the origin to each of the following quadric surfaces. *Hint:* See Example 3 above.

4. $3x^2 + y^2 - 4xz = 4$.

5. $2z^2 + 6xy = 3$.

6. $4y^2 + 2z^2 + 3xy = 18$.

7. Find the largest z for which $2x + 4y = 5$ and $x^2 + z^2 = 2y$.

8. If the temperature at the point (x, y, z) is $T = xyz$, find the hottest point (or points) on the surface of the sphere $x^2 + y^2 + z^2 = 12$, and find the temperature there.

9. The temperature T of the circular plate $x^2 + y^2 \leq 1$ is given by $T = 2x^2 - 3y^2 - 2x$. Find the hottest and coldest points of the plate.

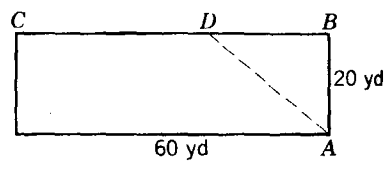
10. The temperature at a point (x, y, z) in the sphere $x^2 + y^2 + z^2 \leq 1$ is given by $T = y^2 + xz$. Find the largest and smallest values which T takes
    (a) on the circle $y = 0$, $x^2 + z^2 = 1$,
    (b) on the surface $x^2 + y^2 + z^2 = 1$,
    (c) in the whole sphere.

11. The temperature of a rectangular plate bounded by the lines $x = \pm 1$, $y = \pm 1$, is given by $T = 2x^2 - 3y^2 - 2x + 10$. Find the hottest and coldest points of the plate.

12. Find the largest and smallest values of the sum of the acute angles that a line through the origin makes with the three coordinate axes.

13. Find the largest and smallest values of the sum of the acute angles that a line through the origin makes with the three coordinate planes.

14. The diagram shows a parking lot, 20 by 60 yd. A contractor has to run a power line from A to C; he can put it on poles around ABC at \$60 a yard or go underground part or all the way at \$75 a yard. If he is honest and wants to minimize the cost, to what point D (if any) should he run the underground part? If he is dishonest and wants the maximize the cost, but will run the line straight along AD (because an inspector is watching), where will D be?



15. Do Example 3b using Lagrange multipliers.

11. CHANGE OF VARIABLES

One important use of partial differentiation is in making changes of variables (for example, from rectangular to polar coordinates). This may give a simpler expression or a simpler differential equation or one more suited to the physical problem one is doing. For example, if you are working with the vibration of a circular membrane, or the flow of heat in a circular cylinder, polar coordinates are better; for a problem about sound waves in a room, rectangular coordinates are better. Consider the following problems.