Math 303: Final Exam

Fall 2006

- Write your name and Student ID number in the space provided below and sign.

Student's Name:	
ID Number:	
Signature:	

- Solve any 5 of the 6 problems. Indicate the problems that you decide not to solve here:

Unsolved Problem:

If you solve all the problems, one with highest points will be dropped.

- You have two and half hours.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the exam within the first 10 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

To be filled by the grader:

Problem	**Points**
1	
2	
3	
4	
5	
6	
Total:	

Problem 1. Find the stationary point(s) of the function

$$f(x,y,z) := \frac{x^2}{2} + y^2 + z^2 + xy - z + \frac{x-y}{4}$$

in the region $V := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 < 1\}$ and determine whether they are minimum, maximum, or saddle point(s). (20 points)

Problem 2. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a twice differentiable function and R be a closed region in $\mathbb{R}^2$ with a smooth boundary C. Let $\hat{n}$ be the unit normal outward vector to C.

2.a) Give the statement of the Green's theorem (in plane). (5 points)

2.b) Use Green's theorem (in plane) to show that the surface integral over R of the Laplacian of ϕ can be expressed as the line integral over C of the directional derivative of ϕ along $\hat{n}$, i.e.,

$$\int\int_R \nabla^2\phi \; dx\, dy = \oint_C D_{\hat{n}}\phi \; ds,$$

where $ds := \sqrt{dx^2 + dy^2}$. (15 points)

Hint: Note that $\hat{n}\; ds = (-dy, dx)$.

Problem 3. Find $y : [0,1] \to \mathbb{R}$ such that $y(0) = 0$, $y(1) = -2$, and the following functional is stationary.

$$\mathcal{F}[y] := \int_0^1 e^{-y(x)} y'(x)^2 \, dx. \qquad \text{(20 points)}$$

Problem 4. Suppose that the differential equation $y^{(4)}(x) + y(x) = \delta(x)$, where $y^{(4)}$ means the forth derivative of y, admits a solution f which has a convergent complex Fourier series: $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{-inx}$.

4.a) Determine the coefficients a_n for all $n \in \mathbb{Z}$. (15 points)

Hint: Use the series representation of $\delta(x)$.

4.b) Express f as a real Fourier series. (5 points)

Problem 5. Let $u : \mathbb{R}^2 \to \mathbb{C}$ be a function and $\tilde{u}(k,t)$ be the Fourier-transform of $u(x,t)$ for each $t \in \mathbb{R}$, i.e.,

$$\tilde{u}(k,x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ikx} dx.$$

Suppose that the following hold.

(i) There is a function $f : \mathbb{R} \to \mathbb{C}$ such that $\tilde{u}(k,t) = f(k)\, e^{-kt^2}$ for all $k \in \mathbb{R}$ and $t \in \mathbb{R}$.

(ii) $u(x,0) = \sin(x)$ for all $x \in \mathbb{R}$.

5.a) Determine $f(k)$ for all $k \in \mathbb{R}$. (10 points)
Hint: Express $u(x,t)$ in terms of $f(k)$ and set $t = 0$.

5.b) Determine $u(x,t)$ for all $(x,t) \in \mathbb{R}^2$. (10 points)

Problem 6. Calculate the (principal value of the) following definite integral. (20 points)

$$I := \int_{-\infty}^{\infty} \frac{e^{-i\pi x}}{(x^2+1)(x-1)}\, dx.$$