Math 303: Quiz # 2

Fall 2008

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 50 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

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1. Find a differential function $f : \mathbb{R}^2 \to \mathbb{R}$ such that both of the following conditions hold.

i) The differential $f(x,y)\,dx + xy\,dy$ is exact.

ii) For all $x \in \mathbb{R}$, $f(x,x) = 0$. (15 points)

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}(xy) = y$$

$$\Rightarrow \quad f(x,y) = \int y\,dy + g(x) = \frac{y^2}{2} + g(x)$$

$$f(x,x) = 0 \Rightarrow \frac{x^2}{2} + g(x) = 0 \Rightarrow g(x) = -\frac{x^2}{2}$$

So $$\boxed{f(x,y) = \frac{y^2 - x^2}{2}}$$

2. Find a real number a such that $f(x-2y)$ is a solution of

$$\frac{\partial^2 \phi}{\partial x^2} + a\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

for every twice differentiable function $f : \mathbb{R} \to \mathbb{R}$. (20 points)

$$\frac{\partial}{\partial x} f(x-2y) = f'(x-2y)$$

$$\frac{\partial}{\partial y} f(x-2y) = -2 f'(x-2y)$$

$$\frac{\partial^2}{\partial x^2} f(x-2y) = f''(x-2y)$$

$$\frac{\partial^2}{\partial x \partial y} f(x-2y) = -2 \frac{\partial}{\partial x} f'(x-2y) = -2 f''(x-2y)$$

$$\frac{\partial^2}{\partial y^2} f(x-2y) = -2 \frac{\partial}{\partial y} f'(x-2y) = 4 f''(x-2y)$$

$$0 = \frac{\partial^2 f(x-2y)}{\partial x^2} + a \frac{\partial^2 f(x-2y)}{\partial x \partial y} + \frac{\partial^2 f(x-2y)}{\partial y^2}$$

$$\Leftrightarrow \quad 0 = f''(x-2y) + a\left(-2 f''(x-2y)\right) + 4 f''(x-2y)$$

$$\Leftrightarrow \quad 0 = (1 - 2a + 4) f''(x-2y)$$

$$\Leftarrow \quad 5 - 2a = 0 \Rightarrow \boxed{a = \frac{5}{2}}$$

3. Consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x, y) = (x^2 - 2y^2)e^{2x-y}$.

a) Show that $(0, 0)$ is a stationary point of g. (10 points)

$g_x = [2x + 2(x^2 - 2y^2)]\, e^{2x-y} = 2(x^2 - 2y^2 + x)\, e^{2x-y}$

$g_y = [-4y - (x^2 - 2y^2)]\, e^{2x-y} = (-x^2 + 2y^2 - 4y)\, e^{2x-y}$

$\Rightarrow g_x(0,0) = 0$ & $g_y(0,0) = 0$ $\Rightarrow$ $(0,0)$ is a stationary point of g.

b) Determine whether $(0, 0)$ is a local minimum, maximum, or saddle point of g. (20 points)

$g_{xx} = 2[2x + 1 + 2(x^2 - 2y^2 + x)]\, e^{2x-y}$

$\Rightarrow g_{xx}(0,0) = 2$

$g_{xy} = 2[-4y - (x^2 - 2y^2 + x)]\, e^{2x-y}$

$\Rightarrow g_{xy}(0,0) = 0$

$g_{yy} = [8y - 4 - (-x^2 + 4y^2 - 4y)]\, e^{2x-y}$

$\Rightarrow g_{yy}(0,0) = -4$

$\Rightarrow$ Hessian at $(0,0)$ = $\begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$ So $(0,0)$ is a saddle point because eigenvalues of the Hessian have opposite sign.

c) Find the other stationary point of g. (10 points)

$g_x = 0 \Rightarrow x^2 - 2y^2 + x = 0$

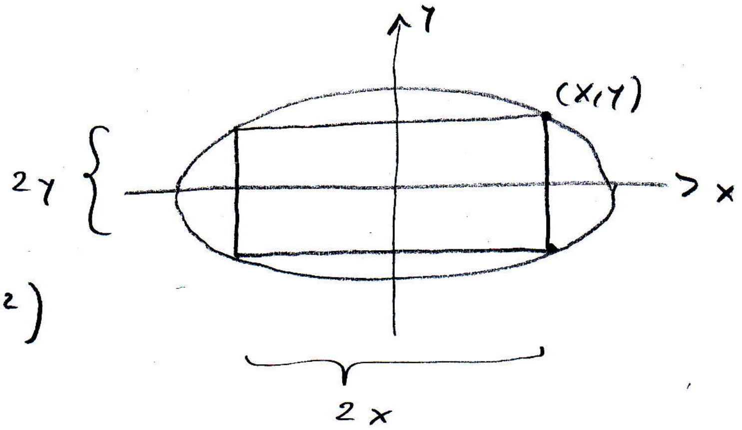
$g_y = 0 \Rightarrow -x^2 + 2y^2 - 4y = 0$

$\xrightarrow{+}$ $x - 4y = 0 \Rightarrow \boxed{x = 4y}$

$\Rightarrow 16y^2 - 2y^2 + 4y = 0 \Rightarrow 2y(7y + 2) = 0$

$\Rightarrow \begin{cases} y = 0 \Rightarrow x = 0 \Rightarrow (0,0) \checkmark \\ y = -\frac{2}{7} \Rightarrow x = -\frac{8}{7} \Rightarrow \boxed{\left(-\frac{8}{7}, -\frac{2}{7}\right)} \end{cases}$

4. Use the method of lagrange multipliers to find the area of the largest rectangle with sides parallel to $x-$ and y-axes that is inscribed in the ellipse defined by $a^2x^2 + b^2y^2 = c^2$, where $a, b, c \in \mathbb{R}^+$. (25 points)



Area: $A = 4xy$

$\phi = a^2x^2 + b^2y^2 - c^2 = 0$

$F = 4xy + \lambda(a^2x^2 + b^2y^2 - c^2)$

$F_x = 4y + 2a^2\lambda x = 0$

$F_y = 4x + 2b^2\lambda y = 0$

$\Rightarrow \quad x \neq 0 \neq y \quad \Rightarrow$

$\lambda = -\dfrac{2y}{a^2x}$

$\lambda = -\dfrac{2x}{b^2y}$

$\Rightarrow \quad \dfrac{y}{a^2x} = \dfrac{x}{b^2y} \quad \Rightarrow \quad \boxed{y^2 = \dfrac{a^2}{b^2}x^2} \quad \Rightarrow \quad y = \dfrac{a}{b}x$

$a^2x^2 + b^2\left(\dfrac{a^2}{b^2}x^2\right) - c^2 = 0$

$\Downarrow$

$2a^2x^2 = c^2 \Rightarrow \boxed{x = \dfrac{c}{\sqrt{2}\,a}}$

$\boxed{y = \dfrac{c}{\sqrt{2}\,b}}$

$\Rightarrow \quad A = 4xy = 4\left(\dfrac{c}{\sqrt{2}\,a}\right)\left(\dfrac{c}{\sqrt{2}\,b}\right)$

$\Rightarrow \quad \boxed{A = \dfrac{2c}{ab}}$