Math 303: Quiz # 3

Fall 2008

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have One hour.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

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1. Give the statement of the divergence theorem in space ($\mathbb{R}^3$). To get full credit you must define all the symbols you use in the formulas you write and explain all the conditions under which these formulas are valid. (20 points)

Let V be a region in $\mathbb{R}^3$ that is bounded by a closed piecewise surface S and $\vec{J} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field that is differentiable on an open subset of $\mathbb{R}^3$ containing V. Then

$$\oiint_S \vec{F} \cdot \hat{n} \, d\sigma = \iiint_V \vec{\nabla} \cdot \vec{F} \, dV$$

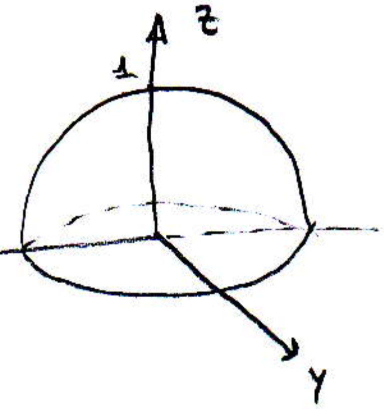
where $\hat{n}$ is the unit normal outward vector to S.

2. Use divergence theorem to evaluate the surface integral $I := \iint_S \mathbf{F}.\mathbf{n}\, d\sigma$, where S is the boundary of the region V in $\mathbb{R}^3$ that is defined by $0 \le z \le \sqrt{1-x^2-y^2}$, $\mathbf{n}$ is the unit normal outward vector to S,

$$\mathbf{F}(x,y,x) = \sin(yz)\mathbf{i} + y^2\mathbf{j} + \sin(\dot{x}y)\mathbf{k}.$$

and $\mathbf{i},\mathbf{j},\mathbf{k}$ are respectively the unit vectors along the x-, y-, and z-axes. (30 points)

$$I = \iiint_V \vec{\nabla}\cdot\vec{F}\, dV$$



$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}\sin(yz) + \frac{\partial}{\partial y} y^2 + \frac{\partial}{\partial z}\sin(xy)$$

$$= 2y$$

use spherical coordinats

On V $r \le 1$, $\theta \in [0, \frac{\pi}{2}]$, $\varphi \in [0, 2\pi)$

$$\Rightarrow \quad I = \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} (2y)\, r^2 \sin\theta \, d\varphi\, d\theta\, dr$$

$$y = r \sin\theta \sin\varphi$$

$$\Rightarrow \quad I = 2 \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} r^3 \sin^2\theta \sin\varphi \, d\varphi\, d\theta\, dr$$

$$= 2 \int_0^1 r^3 dr \int_0^{\pi/2} \sin^2\theta\, d\theta \underbrace{\int_0^{2\pi} \sin\varphi\, d\varphi}_{-\cos\varphi\big|_0^{2\pi} = 0}$$

$$= 0.$$

3. Let $\mathbf{i}$ and $\mathbf{j}$ be the unit vectors along the x- and y-axis in $\mathbb{R}^2$, (r,θ) be the polar coordinates, i.e., $r := \sqrt{x^2+y^2}$ and $\theta := \tan^{-1}(y/x)$, $F_1(r,\theta)$ and $F_2(r,\theta)$ be differentiable functions on the unit disc D defined by $r \le 1$, and $\mathbf{F}(r,\theta) := F_1(r,\theta)\mathbf{i} + F_2(r,\theta)\mathbf{j}$.

3.a) Give the statement of Green's theorem in polar coordinates for D and $\mathbf{F}(r,\theta)$, i.e., express all the quantities appearing in the statement of the Green's theorem in terms of r, θ, $F_1(r,\theta)$, $F_2(r,\theta)$, and their partial derivatives with respect to r and θ. (30 points)

In cartesian coordinates Green's theorem states:

$$\oint_{\partial D} \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1\right) dx\,dy$$

$x = r\cos\theta$, $y = r\sin\theta$

on ∂D : $r = 1$ $\Rightarrow$ $dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$

$$LHS := \oint_{\partial D} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left[-F_1(1,\theta)\sin\theta + F_2(1,\theta)\cos\theta\right] d\theta$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} + \left(-\frac{y}{x^2}\right)\frac{1}{\left(1+\frac{y^2}{x^2}\right)}\frac{\partial}{\partial \theta}$$

$$= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} + \left(\frac{1}{x}\right)\frac{1}{1+\frac{y^2}{x^2}}\frac{\partial}{\partial \theta}$$

$$= \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

$$\Rightarrow \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \cos\theta\frac{\partial F_2}{\partial r} - \frac{\sin\theta}{r}\frac{\partial F_2}{\partial \theta} - \left(\sin\theta\frac{\partial F_1}{\partial r} + \frac{\cos\theta}{r}\frac{\partial F_1}{\partial \theta}\right)$$

$$RHS = \int_0^1\int_0^{2\pi}\left[\cos\theta\frac{\partial F_2}{\partial r} - \frac{\sin\theta}{r}\frac{\partial F_2}{\partial \theta} - \sin\theta\frac{\partial F_1}{\partial r} - \frac{\cos\theta}{r}\frac{\partial F_1}{\partial \theta}\right] r\, d\theta\, dr$$

$$= \int_0^1\int_0^{2\pi}\left[r\cos\theta\frac{\partial F_2}{\partial r} - \sin\theta\frac{\partial F_2}{\partial \theta} - r\sin\theta\frac{\partial F_1}{\partial r} - \cos\theta\frac{\partial F_1}{\partial \theta}\right] d\theta\, dr$$

So Green's thm becomes

$$\int_0^{2\pi}\left(-F_1(1,\theta)\sin\theta + F_2(1,\theta)\cos\theta\right)d\theta = \int_0^1\int_0^{2\pi}\left(r\cos\theta\frac{\partial F_2}{\partial r} - \sin\theta\frac{\partial F_2}{\partial \theta} - r\sin\theta\frac{\partial F_1}{\partial r} - \cos\theta\frac{\partial F_1}{\partial \theta}\right) d\theta\, dr$$

3.b) Use Green's theorem to compute $J := \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$ where D is the unit disc defined by $r \le 1$ and $\mathbf{F}(r,\theta) := e^{-r^2}\sin\theta\,\mathbf{i} - e^{r^2}\cos\theta\,\mathbf{j}$. (20 points)

$$J = \oint \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} [-F_1(1,\theta)\sin\theta + F_2(1,\theta)\cos\theta]\,d\theta$$

$$= \int_0^{2\pi} [-e^{-1}\sin^2\theta + e\cos^2\theta]\,d\theta$$

$$= -e^{-1}\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta + e\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= -e^{-1}\left(\pi - \underbrace{\frac{\sin 2\theta}{4}\Big|_0^{2\pi}}_{0}\right) + e\left(\pi + \underbrace{\frac{\sin 2\theta}{4}\Big|_0^{2\pi}}_{0}\right)$$

$$= \pi\left(e - \frac{1}{e}\right)$$