Math 303: Quiz # 5

Fall 2008

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have one hour.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

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1. Use the series representation of the Dirac Delta function to derive the formula for the coefficients c_n of the complex Fourier series: $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ for a function $f(x)$ in the range $-\pi \leq x < \pi$. (20 points)

$$\delta(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{inx} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-inx}$$

$$f(x) = \int_{-\pi}^{\pi} f(y)\, \delta(y-x)\, dy$$

$$= \int_{-\pi}^{\pi} f(y) \left(\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(x-y)} \right) dy$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iny} f(y)\, dy \right]}_{c_n} e^{inx}$$

$$\Rightarrow \boxed{c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx}$$

2.a) Find an explicit formula for the coefficients c_n of the complex Fourier series: $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ for the function $f(x) := e^x - e^{-x}$ in the range $-\pi \le x < \pi$. (25 points)

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}\,(e^{x} - e^{-x})\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [e^{(-in+1)x} - e^{-(in+1)x}]\,dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(-in+1)x}}{-in+1}\Big|_{-\pi}^{\pi} + \frac{e^{-(in+1)x}}{in+1}\Big|_{-\pi}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{(-in+1)\pi} - e^{-(-in+1)\pi}}{-in+1} + \frac{e^{-(in+1)\pi} - e^{(in+1)\pi}}{in+1}\right]$$

$$\boxed{e^{\pm in\pi} = (-1)^n}$$

$$= \frac{(-1)^n}{2\pi}\left[\frac{(in+1)(e^{\pi} - e^{-\pi}) + (-in+1)(e^{-\pi} - e^{\pi})}{n^2+1}\right]$$

$$= \left(\frac{(-1)^n(e^{\pi} - e^{-\pi})}{n^2+1}\right)(in+1+in-1)$$

$$= \frac{2in(-1)^n(e^{\pi} - e^{-\pi})}{n^2+1}$$

2.b) Find an explicit formula for the coefficients a_n and b_n of the real Fourier series:

$$\frac{a_0}{2} + \sum_{n=-\infty}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

for the function $f(x) := e^x - e^{-x}$ in the range $-\pi \le x < \pi$. (15 points)

Note: You can do this most easily using your response to part **2.a)**.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + c_{-n} e^{-inx})$$

$$= c_0 + \sum_{n=1}^{\infty} [c_n(\cos(nx) + i\sin(nx)) + c_{-n}(\cos(nx) - i\sin(nx))]$$

$$= \underbrace{c_0}_{\frac{a_0}{2}} + \sum_{n=1}^{\infty} [\underbrace{(c_n + c_{-n})}_{a_n}\cos(nx) + \underbrace{i(c_n - c_{-n})}_{b_n}\sin(nx)]$$

$\Rightarrow \quad a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n})$

Because $f(x)$ is odd, $\boxed{a_n = 0, \quad \forall n \geq 0.}$

$$b_n = i\left[\frac{2in(-1)^n(e^{\pi} - e^{-\pi})}{n^2+1} - \frac{2i(-n)(-1)^{-n}(e^{\pi} - e^{-\pi})}{n^2+1}\right]$$

$$= \left[\frac{2n(-1)^n(e^{\pi} - e^{-\pi})}{n^2+1}\right]\left[-1-1\right]$$

$$\boxed{b_n = \frac{4n(-1)^n(e^{\pi} - e^{-\pi})}{n^2+1}, \quad \forall n \geq 1}$$

3. Let $\tilde{f}(k)$ denote the Fourier transform of a function $f(x)$.

3.a) Use the definition of the Fourier transform and properties of the Dirac Delta function to express the inverse Fourier transform of $\tilde{f}(2k) - \tilde{f}(-k)$ in terms of f. (30 points)

let $g(x) := \mathcal{F}^{-1}\{\tilde{f}(2k) - \tilde{f}(-k)\} \Rightarrow$

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\tilde{f}(2k) - \tilde{f}(-k)]\, e^{ikx}\, dk$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{-iky}\, dy$$

$$\Rightarrow g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} [f(y)\, e^{-2iky} - f(y)\, e^{iky}]\, dy \right) e^{ikx}\, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \underbrace{\left(\int_{-\infty}^{\infty} [e^{i(x-2y)k} - e^{i(x+y)k}]\, dk \right)}_{2\pi\,\delta(x-2y) - 2\pi\,\delta(x+y)} dy$$

$$= \int_{-\infty}^{\infty} f(y) \underbrace{\delta(x-2y)}_{\delta(2y-x) = \frac{1}{2}\delta(y-\frac{x}{2})}\, dy - \int_{-\infty}^{\infty} f(y) \underbrace{\delta(x+y)}_{\delta(y-(-x))}\, dy$$

$$= \frac{1}{2} f\left(\frac{x}{2}\right) - f(-x)$$

3.b) Find the inverse Fourier transform of $\tilde{f}(2k) - \tilde{f}(-k)$ for $f(x) = x\, e^{-x^4}$. (10 points)

$$\mathcal{F}^{-1}\{\tilde{f}(2k) - \tilde{f}(-k)\} = \frac{1}{2}\left[\frac{x}{2}\, e^{-\frac{x^4}{16}} + x\, e^{-x^4}\right]$$

$$= \frac{x}{4}\left(e^{-\frac{x^4}{16}} + 2\, e^{-x^4}\right)$$