Math 303: Quiz # 6

Fall 2008

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 50 minutes.
- You may use any statement which has been proven in class, except for the cases where you are asked to reproduce the proof of that statement.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 5 points will be deduced from your grade (You may or may not get an answer to your question(s).)

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1.a) Give the definition of a pole of a function $f : \mathbb{C} \to \mathbb{C}$. (5 points)

$z_0 \in \mathbb{C}$ is said to be a pole of f if it is a singular point of f and the principal part of the Laurent series expansion of f about z_0 is a finite sum (has finitely many terms.)

1.b) Give the definition of the residue of a function $f : \mathbb{C} \to \mathbb{C}$ at a singular point. (5 points)

Residue of f at a singular point z_0 is the coefficient of the term $\frac{1}{z-z_0}$ in the Laurent series expansion of f about z_0.

1.c) Give the statement of Cauchy's theorem (on contour integrals.) (10 points)

Let $f: \mathbb{C} \to \mathbb{C}$ be a function that is analytic in a region bounded by a contour C and on C.

Then $\oint_C f(z)\, dz = 0$.

2. Find the principal part of the Laurent series expansion of $\frac{\sin z}{(e^z-1)^3}$ about $z=0$. (30 points)

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} \pm \cdots = z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} \pm \cdots\right)$$

$$e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} \pm \cdots = z\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)$$

$$\Rightarrow \quad \frac{\sin z}{(e^z-1)^3} = \frac{z\left(1 - \frac{z^2}{3!} \pm \cdots\right)}{z^3\left(1 + \frac{z}{2!} + \cdots\right)^3}$$

This suggests that $z=0$ is a pole of order 2.

$$\lim_{z\to 0} z^2 \frac{\sin z}{(e^z-1)^3} = \lim_{z\to 0} \frac{1 - \frac{z^2}{3!} \pm \cdots}{\left(1 + \frac{z}{2!} + \cdots\right)^3} = 1$$

$\Rightarrow$ $z=0$ is a pole of order 2 and $\boxed{b_2 = 1}$

$$\boxed{\text{The principal part} = \frac{b_1}{z} + \frac{b_2}{z^2}}$$

$b_1 = \operatorname{Res}(0) = g'(0)$ where $g(z) = \frac{z^2 \sin z}{(e^z-1)^3} = \frac{1 - \frac{z^2}{3!} \pm \cdots}{\left(1 + \frac{z}{2!} + \cdots\right)^3}$

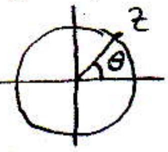
$$g'(z) = \frac{\left(-\frac{2z}{3!} \pm \cdots\right)\left(1 + \frac{z}{2!} + \cdots\right)^3 - 3\left(1 + \frac{z}{2!} + \cdots\right)^2 \frac{1}{2!}\left(1 - \frac{z^2}{3!} \pm \cdots\right)}{\left(1 + \frac{z}{2} + \cdots\right)^6}$$

$\Downarrow$

$$b_1 = g'(0) = \frac{0 - \frac{3}{2!}}{1} \Rightarrow \boxed{b_1 = -\frac{3}{2}}$$

$$\Rightarrow \boxed{\text{The principal part} = -\frac{3}{2z} + \frac{1}{z^2}}$$

3.a) Find a contour C and a function $f : \mathbb{C} \to \mathbb{C}$ such that $\displaystyle\int_0^{2\pi} \frac{d\theta}{2 - \sin\theta} = \oint_C f(z)\, dz$. (10 points)



Let C be the unit circle: $|z| = 1$.

$\Rightarrow \quad z = e^{i\vartheta}, \quad dz = ie^{i\vartheta} d\vartheta \Rightarrow d\vartheta = \dfrac{dz}{iz}$

$$\sin\theta = \frac{z - z^{-1}}{2i} \Rightarrow f(z) := \frac{1}{iz\left[2 - \frac{z - \frac{1}{z}}{2i}\right]} = \frac{2}{-z^2 + 4iz + 1}.$$

3.b) Determine the singular point(s) of $f(z)$ that lie in the region bounded by C and compute the residue of $f(z)$ at these points. (35 points)

$$-z^2 + 4iz + 1 = 0 \Rightarrow \underbrace{-(z^2 - 4iz - 1)}_{(z-2i)^2 + 3} = 0 \;\hookrightarrow\; z = 2i \pm \sqrt{3}\, i$$

$$\Rightarrow \quad z = i\left[2 \pm \sqrt{3}\right] \qquad \sqrt{3} \sim 1.7$$

$$\Rightarrow |(2+\sqrt{3})i| = |2+\sqrt{3}| > 1 \quad \& \quad |(2-\sqrt{3})i| = |2-\sqrt{3}| < 1$$

So $(2-\sqrt{3})i$ is the singular point lying in this region.

$$f(z) = \frac{2}{-[z-(2-\sqrt{3})i][z-(2+\sqrt{3})i]}$$

$$\lim_{z \to 2-\sqrt{3}} [z - (2-\sqrt{3})i]\, f(z) = \frac{2}{-(2-\sqrt{3}-2-\sqrt{3})i} = \frac{2}{2\sqrt{3}i} = \frac{1}{\sqrt{3}i}$$

$\Rightarrow$ $2-\sqrt{3}$ is a simple pole & $\mathrm{Res}(2-\sqrt{3}) = \dfrac{1}{i\sqrt{3}} = -\dfrac{i}{\sqrt{3}}$

3.c) Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{2 - \sin\theta}$. (5 points)

$$\int_0^{2\pi} \frac{d\vartheta}{2 - \sin\vartheta} = 2\pi i\, \mathrm{Re}((2-\sqrt{3})i) = 2\pi i\left(\frac{-i}{\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}.$$