Math 303: Final Exam

January 03, 2018

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 2.5 hours.
- You must show the details of all your work. Illegible and ambiguous explanations and calculations will lead to deductions from your grade.

Problem 1 (10 points) Let $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{B} : \mathbb{R}^3 \to \mathbb{R}^3$ be vector fields having first partial derivatives. Use the Levi Civita symbol to determine $\alpha, \beta \in \mathbb{R}$ such that

$$\mathbf{A} \cdot (\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A})) = \alpha\, \mathbf{A} \cdot [(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A}] + \beta\, \mathbf{B} \cdot [(\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{A}].$$

$$\mathbb{A}\cdot(\mathbb{B}\times(\nabla\times\mathbb{A})) = \sum_{i=1}^{3} A_i \sum_{j,k=1}^{3} \epsilon_{ijk} B_j \sum_{l,m=1}^{3} \epsilon_{klm} \partial_l A_m$$

$$= \sum_{i,j,k,l,m=1}^{3} A_i \underbrace{\epsilon_{ijk}\epsilon_{klm}}_{\epsilon_{kij}\epsilon_{klm}} B_j \partial_l A_m$$

$$= \sum_{i,j,l,m=1}^{3} A_i (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) B_j \partial_l A_m$$

$$= \sum_{i,j=1}^{3} (A_i B_j \partial_i A_j - A_i B_j \partial_j A_i)$$

$$= \sum_{i,j=1}^{3} B_j A_i \partial_i A_j - \sum_{i,j=1}^{3} A_i B_j \partial_j A_i$$

$$= \mathbb{B}\cdot((\mathbb{A}\cdot\nabla)\mathbb{A}) - \mathbb{A}\cdot((\mathbb{B}\cdot\nabla)\mathbb{A})$$

$$\Rightarrow \boxed{\alpha = -1, \quad \beta = 1}$$

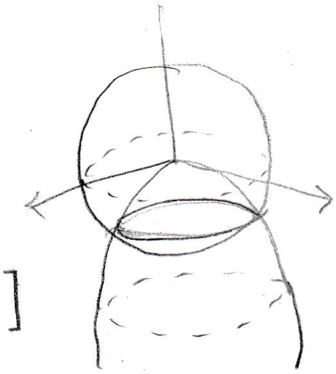
Problem 2 (20 points) Let a and r be positive real numbers,

$$\mathbf{A}(x,y,z) := (yz^2, -xz, e^{xyz}),$$

S_1 and S_2 be respectively the surfaces defined by $z = -\alpha(x^2+y^2)$ and $x^2+y^2+z^2 = r^2$, and S be the part of S_2 that lies above S_1, i.e.,

$$S := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = r^2 \text{ and } z \geq -\alpha(x^2+y^2)\}.$$

Use Stokes' theorem to evaluate the surface integral $I := \iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, dS$, where $\mathbf{n}$ is the unit normal outward vector for S.



$$\partial S = S_1 \cap S_2 : \quad x^2+y^2+z^2 = r^2, \quad z = -\alpha(x^2+y^2)$$

$$\Rightarrow \quad -\frac{z}{\alpha} + z^2 = r^2$$

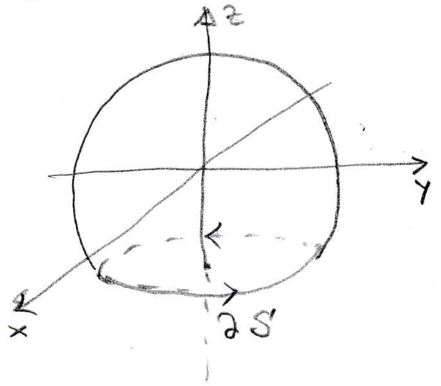
$$\Rightarrow \quad z^2 - \frac{z}{\alpha} - r^2 = 0 \Rightarrow z = \frac{1}{2}\left[\frac{1}{\alpha} \pm \sqrt{\frac{1}{\alpha^2} + 4r^2}\right]$$

$$z < 0 \quad \Rightarrow \quad \boxed{z = \frac{1}{2\alpha}\left(1 - \sqrt{1+4\alpha^2 r^2}\right)}$$

$$\Rightarrow \quad \boxed{x^2+y^2 = -\frac{1}{2\alpha^2}\left(1-\sqrt{1+4\alpha^2r^2}\right) = \frac{1}{2\alpha^2}\left(\sqrt{1+4\alpha^2r^2} - 1\right)}$$

Let $R := \frac{1}{\sqrt{2}\,\alpha}\sqrt{\left(\sqrt{1+4\alpha^2r^2} - 1\right)}$ so that

$$\boxed{x^2+y^2 = R^2}$$



$$\vec{r} \in \partial S \Rightarrow \quad \vec{r} = (\overbrace{R\cos\varphi}^{x}, \overbrace{R\sin\varphi}^{y}, \overbrace{-\sqrt{r^2-R^2}}^{z})$$

$$\Rightarrow \quad d\vec{r} = (-R\sin\varphi\, d\varphi, R\cos\varphi\, d\varphi, 0)$$

$$I = \oint_{\partial S} \vec{A} \cdot d\vec{r} = \int_0^{2\pi} \left[(R\sin\varphi)(r^2-R^2)(-R\sin\varphi)\, d\varphi + (R\cos\varphi)\sqrt{r^2-R^2}\,(R\cos\varphi)\, d\varphi\right]$$

$$= R^2\left[(r^2-R^2)\underbrace{\left(-\int_0^{2\pi} \underbrace{\sin^2\varphi}_{\frac{1-\cos 2\varphi}{2}}\, d\varphi\right)}_{-\pi} + \sqrt{r^2-R^2}\underbrace{\int_0^{2\pi} \underbrace{\cos^2\varphi}_{\frac{1+\cos 2\varphi}{2}}\, d\varphi}_{\pi}\right]$$

$$= \pi R^2\left(R^2 - r^2 + \sqrt{r^2-R^2}\right)$$

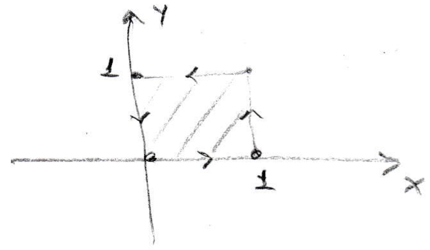
Problem 3 Let D be the region in $\mathbb{R}^2$ that is bounded by the square S with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. Green's Theorem states that for any vector-valued function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ that is differentiable in D and on S

$$\oint_S \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy,$$

where $\mathbf{r} =: (x, y)$ and $\mathbf{F} =: (F_1, F_2)$.

3.a (10 points) Prove this statement by computing both sides of the above equation.

$$\text{LHS} := \oint_S \vec{F}\cdot d\vec{r} = \int_0^1 F_1(x,0)\,dx + \int_0^1 F_2(1,y)\,dy + \int_1^0 F_1(x,1)\,dx + \int_1^0 F_2(0,y)\,dy$$



$$= \int_0^1 [F_1(x,0) - F(x,1)]\,dx + \int_0^1 [F_2(1,y) - F_2(0,y)]\,dy$$

$$\text{RHS} := \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

$$= \int_0^1\int_0^1 \frac{\partial F_2}{\partial x}\,dx\,dy - \int_0^1\int_0^1 \frac{\partial F_1}{\partial y}\,dy\,dx$$

$$= \int_0^1 F_2(x,y)\Big|_{x=0}^{x=1} dy - \int_0^1 F_1(x,y)\Big|_{y=0}^{y=1} dx$$

$$= \int_0^1 (F_2(1,y) - F_2(0,y))\,dy \underbrace{- \int_0^1 [F_1(x,1) - F_1(x,0)]\,dx}_{+\int_0^1 [F_1(x,0) - F_1(x,1)]\,dx}$$

$$= \text{LHS}. \quad \boxtimes$$

3.b (10 points) Let C be the contour defined by giving counter-clockwise orientation to the square S. Prove Cauchy's theorem for C, i.e., show that for any functions $f : \mathbb{C} \to \mathbb{C}$ that is holomorphic in the region D bounded by C, $\oint_C f(z)dz = 0$.

$$\oint_C f(z)\,dz = \oint_C [u(x,y) + i v(x,y)](dx + i\,dy)$$

where $u := Re(f)$, $v := Im(f)$, $x := Re(z)$, $y := Im(z)$

$$\Rightarrow \oint_C f(z)\,dz = \underbrace{\oint_C [u(x,y)\,dx - v(x,y)\,dy]}_{\vec{F}\cdot d\vec{r}} + i \underbrace{\oint_C [u(x,y)\,dy + v(x,y)\,dx]}_{\vec{G}\cdot d\vec{r}}$$

where
$$\vec{F}(x,y) = (u(x,y), -v(x,y))$$
$$\vec{G}(x,y) = (v(x,y), u(x,y))$$

Applying Green's thm $\Rightarrow$

$$\oint_C f(z)\,dz = \iint_D \underbrace{\left[\frac{\partial}{\partial x}(-v) - \frac{\partial}{\partial y}(u)\right]}_{\underbrace{-\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)}_{0}} dx\,dy + i \iint_D \underbrace{\left[\frac{\partial}{\partial x}(u) - \frac{\partial}{\partial y}(v)\right]}_{\underbrace{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}}_{0}} dx\,dy$$

$\Leftarrow$ By Cauchy-Riemann Conditions $\Rightarrow$

$= 0$

Problem 4 (25 points) Evaluate the (principal value) of the improper integral:

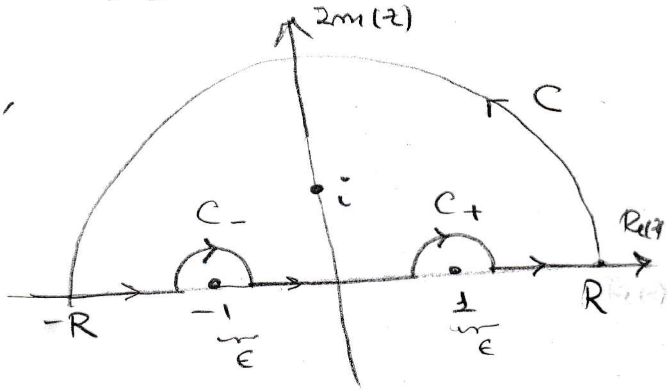
$$I := \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^4-1}\,dx,$$

for all $k \in \mathbb{R}$.

For $k<0$, $I = \int_{-\infty}^{\infty} \frac{e^{-i|k|x}}{x^4-1}\,dx$ $\quad x' = -x \Rightarrow dx = -dx'$

$$= \int_{-\infty}^{\infty} \frac{e^{i|k|x'}}{x'^4-1}\,dx'$$

$$\Rightarrow \forall k \in \mathbb{R}, \quad I = \int_{-\infty}^{\infty} \frac{e^{i|k|x}}{x^4-1}\,dx$$



$$I_C = \oint_C \underbrace{\frac{e^{i|k|z}}{z^4-1}}_{f(z)}\,dz = 2\pi i\, R(i)$$

$z^4-1=0 \Rightarrow z = \pm 1, \quad z = \pm i$

$$I = \lim_{\epsilon\to 0}(I_C - I_- - I_+) \qquad I_\pm = \int_{C_\pm} f(z)\,dz$$

$$R(i) = \lim_{z\to i} \frac{e^{i|k|z}}{(z+i)(z^2-1)} = \frac{e^{-|k|}}{2i(-2)} = -\frac{e^{-|k|}}{4i} \Rightarrow \boxed{I_C = -\frac{\pi e^{-|k|}}{2}}$$

C_-: $\quad z = -1+\epsilon e^{i\theta}, \quad dz = i\epsilon e^{i\theta}\,d\theta$

$$I_- = \int_{\pi}^{0} \frac{e^{i|k|(-1+\epsilon e^{i\theta})}}{(-1+\epsilon e^{i\theta})^4-1}\cdot i\epsilon e^{i\theta}\,d\theta$$

$$\lim_{\epsilon\to 0} I_- = \frac{-\pi i e^{-i|k|}}{-4} = \frac{i\pi e^{-i|k|}}{4}$$

C_+: $\quad z = 1+\epsilon e^{i\theta}, \quad dz = i\epsilon e^{i\theta}\,d\theta$

$$I_+ = \int_{\pi}^{0} \frac{e^{i|k|(1+\epsilon e^{i\theta})}}{(1+\epsilon e^{i\theta})^4-1}\, i\epsilon e^{i\theta}\,d\theta$$

$$\lim_{\epsilon\to 0} I_+ = \frac{-\pi i e^{i|k|}}{4} = \frac{-i\pi e^{i|k|}}{4}$$

$$\Rightarrow I = -\frac{\pi e^{-|k|}}{2} - \frac{\pi}{4}\underbrace{\left(ie^{-i|k|} - ie^{i|k|}\right)}_{-i\,(2i)\sin(|k|)} = -\frac{\pi e^{-|k|}}{2} - \frac{\pi}{2}\sin(|k|)$$

$$\Rightarrow \boxed{I = -\frac{\pi}{2}\left(e^{-|k|} + \sin|k|\right)}$$

Problem 5 (15 points) Use Fourier transformation to find a particular solution for the differential equation $\frac{d^4}{dx^4}y(x) - y(x) = \delta(x)$, where $\delta(x)$ is the Dirac delta function.

Hint: Evaluate Fourier transform of both sides of this equation to find the Fourier transform of a solution. Then use your response to Problem 4 to find the solution.

$$\mathcal{F}\{y^{(4)}(x) - y(x)\} = \mathcal{F}\{\delta(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,\delta(x)\,dx$$

$$(ik)^4\,\tilde{Y}(k) - \tilde{Y}(k) = \frac{1}{\sqrt{2\pi}}$$

$$\Rightarrow \quad \tilde{Y}(k) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{k^4-1}$$

$$\Downarrow$$

$$Y(x) = \mathcal{F}^{-1}\{\tilde{Y}(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,\frac{1}{\sqrt{2\pi}}\,\frac{1}{k^4-1}\,dk$$

$$= \frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty}\frac{e^{ikx}}{k^4-1}\,dk}_{-\frac{\pi}{2}\left(e^{-|x|} + \sin|x|\right)}$$

$$\Rightarrow \quad \boxed{Y(x) = -\frac{1}{4}\left(e^{-|x|} + \sin|x|\right)}$$

in $(-\pi, \pi)$

Problem 6 (10 points) Find the real Fourier series for $\delta(x^2 - 4)$ where $\delta(x)$ is the Dirac delta function.

$$f(x) = x^2 - 4 \qquad f(x) = 0 \Rightarrow x = \pm 2$$

$$f'(x) = 2x \qquad f'(\pm 2) = \pm 4$$

$$\Rightarrow \delta(x^2 - 4) = \frac{\delta(x-2)}{|f'(2)|} + \frac{\delta(x+2)}{|f'(-2)|}$$

$$= \frac{1}{4}\left[\delta(x-2) + \delta(x+2)\right]$$

$$\delta(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{inx}$$

$$\Rightarrow \delta(x^2-4) = \frac{1}{8\pi}\sum_{n=-\infty}^{\infty} \underbrace{\left(e^{in(x-2)} + e^{in(x+2)}\right)}_{\underbrace{(e^{-2in} + e^{2in})}_{2\cos(2n)} e^{inx}}$$

$$= \frac{1}{4\pi}\sum_{n=-\infty}^{\infty} \cos(2n)\, e^{inx}$$

Because $\delta(x^2-4)$ is even $\Rightarrow$ $\delta(x^2-4) = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty} \cos(2n)\, e^{-inx}$

$$\Rightarrow \delta(x^2-4) = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty} \cos(2n) \underbrace{\left(\frac{e^{inx} + e^{-inx}}{2}\right)}_{\cos(nx)}$$

$$= \frac{1}{4\pi} + \frac{1}{4\pi}\sum_{n=1}^{\infty} 2\cos(2n)\cos(nx)$$

$$\Rightarrow \boxed{\delta(x^2-4) = \frac{1}{4\pi} + \frac{1}{2\pi}\sum_{n=1}^{\infty} \cos(2n)\cos(nx)}$$