Math 303: Quiz # 3

Fall 2017

- Write your name and Student ID number in the space provided below and sign.

Name, Last Name:	
ID Number:	
Signature:	

- You have 45 minutes.
- **Give details of your response to each problem. You will not be given any credit, if it is not clear how you have obtained your response.**

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1 (10 points) Find the residue of $f(z) := \dfrac{\cos^2 z}{(z-\frac{\pi}{2})^4}$ at $\frac{\pi}{2}$.

$$\lim_{z\to\frac{\pi}{2}} (z-\tfrac{\pi}{2}) f(z) = \lim_{z\to\frac{\pi}{2}} \frac{\cos^2 z}{(z-\frac{\pi}{2})^3} = \lim_{z\to\frac{\pi}{2}} \frac{-2\sin\theta\cos\theta}{3(z-\frac{\pi}{2})^2} = \lim_{z\to\frac{\pi}{2}} \frac{-\sin 2\theta}{3(z-\frac{\pi}{2})^2}$$

$$= \lim_{z\to\frac{\pi}{2}} \frac{-2\cos 2\theta}{6(z-\frac{\pi}{2})} = \infty \quad \text{Not a simple pole}$$

$$\lim_{z\to\frac{\pi}{2}} (z-\tfrac{\pi}{2})^2 f(z) = \lim_{z\to\frac{\pi}{2}} \frac{\cos^2 z}{(z-\frac{\pi}{2})^2} = \lim_{z\to\frac{\pi}{2}} \frac{-\sin 2\theta}{2(z-\frac{\pi}{2})} = \lim_{z\to\frac{\pi}{2}} \frac{-2\cos 2\theta}{2} = 1$$

$\Rightarrow$ $\frac{\pi}{2}$ is a double pole

$$R(\tfrac{\pi}{2}) = \lim_{z\to\frac{\pi}{2}} g'(z) \qquad \text{where } g(z) = (z-\tfrac{\pi}{2})^2 f(z) = \frac{\cos^2 z}{(z-\frac{\pi}{2})^2}$$

$$\Rightarrow g'(z) = \frac{-2\sin z\cos z\,(z-\frac{\pi}{2})^2 - 2(z-\frac{\pi}{2})\cos^2 z}{(z-\frac{\pi}{2})^4} = \frac{-\sin(2z)(z-\frac{\pi}{2}) - 2\cos^2 z}{(z-\frac{\pi}{2})^3}$$

$$\lim_{z\to\frac{\pi}{2}} g'(z) = \lim_{z\to\frac{\pi}{2}} \frac{-2\cos(2z)(z-\frac{\pi}{2}) - \sin(2z) + 4\sin z\cos z}{3(z-\frac{\pi}{2})^2}$$

$$= \lim_{z\to\frac{\pi}{2}} \frac{-2\cos(2z)(z-\frac{\pi}{2}) + \sin(2z)}{3(z-\frac{\pi}{2})^2}$$

$$= \lim_{z\to\frac{\pi}{2}} \frac{4\sin(2z)(z-\frac{\pi}{2}) - 2\cos(2z) + 2\cos(2z)}{6(z-\frac{\pi}{2})} = 0$$

$$\Rightarrow \boxed{R(\tfrac{\pi}{2}) = 0}$$

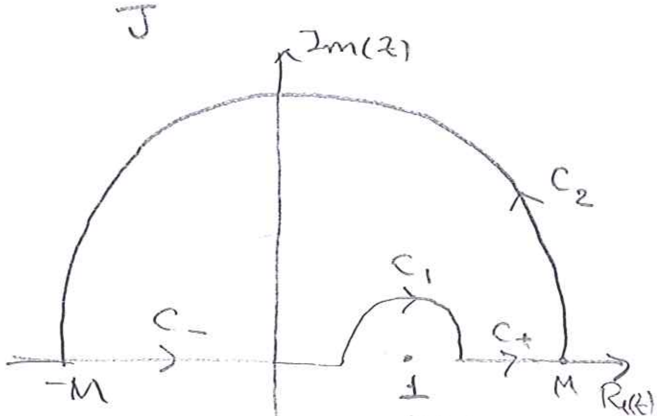
2 (30 points) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(\pi x)}{(x^2+1)(x-1)}\,dx$.

Hint: You may use the fact that $\sin(\pi x) = \mathrm{Im}(e^{i\pi x})$ for $x \in \mathbb{R}$.

$$I := \int_{-\infty}^{\infty} \frac{\sin(\pi x)}{(x^2+1)(x-1)}\,dx = \mathrm{Im} \underbrace{\int_{-\infty}^{\infty} \frac{e^{i\pi x}}{(x^2+1)(x-1)}\,dx}_{J}$$

$$f(z) := \frac{e^{i\pi z}}{(z^2+1)(z-1)}$$

$$C := C_- \sqcup C_1 \sqcup C_+ \sqcup C_2$$



On C_2: $z = Me^{i\theta}$ $\quad \theta \in [0, \pi]$

$\Downarrow \qquad\qquad \Downarrow$

$dz = iMe^{i\theta}d\theta \qquad \sin\theta \geq 0$

$$\Rightarrow \lim_{M\to\infty} \int_{C_2} f(z)\,dz = \lim_{M\to\infty} \int_0^{\pi} \frac{e^{i\pi Me^{i\theta}}\, iMe^{i\theta}\,d\theta}{(M^2e^{2i\theta}+1)(Me^{i\theta}-1)} = 0$$

because $|e^{i\pi Me^{i\theta}}| = e^{-\pi M\sin\theta}$ & $\sin\theta \geq 0$.

On C_1: $z = 1 + \epsilon e^{i\theta}$ $\quad dz = i\epsilon e^{i\theta}d\theta$

$$\Rightarrow \lim_{\epsilon\to 0} \int_{C_1} f(z)\,dz = \lim_{\epsilon\to 0} \int_{\pi}^{0} \frac{e^{i\pi(1+\epsilon e^{i\theta})}\, i\epsilon e^{i\theta}\,d\theta}{[(1+\epsilon e^{i\theta})^2+1]\,\epsilon e^{i\theta}} = \frac{i\overset{\nearrow -1}{e^{i\pi}}}{2} \underbrace{\int_{\pi}^{0} d\theta}_{-\pi} = \frac{i\pi}{2}$$

$$\oint_C f(z)\,dz = 2\pi i R(i)$$

$$\lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{e^{i\pi z}}{(z+i)(z-1)} = \frac{e^{-\pi}}{2i(i-1)} = \frac{e^{-\pi}}{2(-1-i)} = \frac{-e^{-\pi}(1-i)}{4}$$

So $z = i$ is a simple pole & $R(i) = \dfrac{-e^{-\pi}(1-i)}{4}$

$$\Rightarrow J = \lim_{\substack{M\to\infty\\ \epsilon\to 0}} \Big[\oint_C f(z)\,dz - \overset{\to i\pi/2}{\int_{C_1} f(z)\,dz} - \overset{\to 0}{\int_{C_2} f(z)\,dz}\Big] = 2\pi i\left(\frac{-e^{-\pi}(1-i)}{4}\right) - \frac{i\pi}{2}$$

$$= \frac{\pi}{2}\left[-e^{-\pi} + i(-e^{-\pi} - 1)\right] = -\frac{\pi}{2}\left[e^{-\pi} + i(1+e^{-\pi})\right]$$

$$\boxed{I = \mathrm{Im}(J) = -\frac{\pi}{2}(1+e^{-\pi})}$$