A History of the Divergence, Green's, and Stokes' Theorems

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$$\iiint\limits_{T} \operatorname{div} \mathbf{F} dV = \iint\limits_{\partial T} \mathbf{F} \cdot \mathbf{n}\, dA$$

$$\iint\limits_{R} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint\limits_{\partial R} (F_1 dx + F_2 dy)$$

$$\iint\limits_{S} (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dA = \oint\limits_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

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1 Prologue

Mathematics has always had many great minds contributing to form something that is truly amazing. In the beginning of mathematics, cultures were much more closed off than they are today. As time went on, cultures realized how beneficial the sharing of ideas could be. In the 17th, 18th, and 19th centuries, countries were becoming much more open to collaboration which allowed for a revolution of new ideas. Once calculus was discovered, new doors for mathematics were opened. In the 18th century, the Divergence Theorem was proposed. This theorem "equates a surface integral with a triple integral over the volume inside the surface" [6]. In the following century it would be proved along with two other important theorems, known as Green's Theorem and Stokes' Theorem. Green's Theorem can be described as the two-dimensional case of the Divergence Theorem, while Stokes' Theorem is a general case of both the Divergence Theorem and Green's Theorem. Overall, once these theorems were discovered, they allowed for several great advances in science and mathematics which are still of grand importance today.

2 The Divergence Theorem

2.1 History of the Divergence Theorem

The origins of applied mathematics can be traced all the way back to a man named Joseph-Louis Lagrange. The work Lagrange started in the 18th century was made possible because of the mathematicians before him such as Isaac Newton with his discovery of calculus. At the age of nineteen, Lagrange sent his work on calculus of variations to Leonhard Euler in 1755 [11]. Euler had written back explaining how impressed he was with his results. Lagrange was then appointed professor of mathematics at the Royal Artillery School about one month later [11]. His great talent and original ideas were already being noticed by several well-known mathematicians. It was not long before Lagrange was applying the calculus of variations to mechanics and gained even more popularity in the mathematical and scientific worlds. In 1757, he was a leading founder of a new society called the Royal Academy of Sciences of Turin [11]. One of the main goals of the society was to publish articles in the *Mélanges de Turin* which translates to "mixture of Turin" [11]. Lagrange contributed greatly to the first three volumes of this journal. He then began working in differential equations and various applications of mathematics such as fluid mechanics [11]. In 1764, he discovered what would be known as the Divergence Theorem [15]. Although he did not provide a proof for this theorem, he would go on to formulate a great many other works. The Divergence Theorem would take much more manpower to finally bring forth a proof. The men who would make the most notable advances were mathematicians such as Karl Friedrich Gauss, George Green, and Mikhail Vasilyevich Ostrogradsky.

The Divergence Theorem would have no more progress until a man named Karl Friedrich Gauss rediscovered it in 1813 [14]. As a child, Gauss was known to have extraordinary talent. He is known for summing the integers 1 to 100 at a very young age in elementary school [8]. Gauss would be the first to inscribe a seventeen-gon and at the age of only nineteen [8]. He published his discovery in the *Disquisitiones Arithmeticae* or "number research" [8]. In 1799, Gauss received his degree from the Brunswick Collegium Carolinum [8]. He then earned his doctorate at the University of Helmstedt with his submission of the Fundamental Theorem of Algebra [8]. Gauss would then go on to make significant advances in the Divergence Theorem and its special case now known as Green's Theorem [8]. In 1813, Gauss formulated Green's Theorem, but could not provide a proof [14]. Although Gauss did excellent work, he would not publish his results until 1833 and 1839 [2]. This would, in fact, be too late to receive proper credit as the Russian Mikhail Vasilyevich Ostrogradsky would be the first to prove the Divergence Theorem 1831 [2]. Another mathematician, George Green, rediscovered the Divergence Theorem,without knowing of the work Lagrange and Gauss [15]. Green published his work in 1828, but those who read his results could not thoroughly understand his work, and thus nearly discarded it. His work contained the two-dimensional case of the Divergence Theorem, Green's Theorem.

On September 24, 1801, Ostrogradsky was born [12]. In 1816, he studied physics and mathematics at the University of Kharkov [12]. However, Ostrogradsky never received his degree due to religious and internal problems [12]. Instead he headed to Paris and studied under several great mathematicians, such as Pierre-Simon Laplace, Joseph Fourier, and Augustin-Louis Cauchy [12]. In 1831, he rediscovered the Divergence Theorem and provided a proof. Finally, the theorem was proved.

2.2 A Proof of the Divergence Theorem

The Divergence Theorem. *Let T be a subset of $\mathbb{R}^3$ that is compact with a piecewise smooth boundary. Now let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector-valued function with continuous first partial derivatives defined on a neighborhood of T, ∂T. Then*

$$\iiint_T \operatorname{div} \mathbf{F} dV = \iint_{\partial T} \mathbf{F} \cdot \mathbf{n} \, dA,$$

where $\mathbf{n}$ is normal, or perpendicular, to the surface ∂T, and where V is the volume of T and A is the area of ∂T.

Proof. Proving this theorem for a rectangular parallelepiped will in fact prove the theorem for any arbitrary surface, as the nature of the Riemann sums of the triple integral ensures this.

Let $T = \{(x, y, z) | \; x_1 < x < x_2, \; y_1 < y < y_2, \; z_1 < z < z_2\}$ with ∂T outwardly orientated, and let the sides T_1 and T_2 of T be perpendicular to the x-axis, the sides T_3 and T_4 of T be perpendicular to the y-axis, and the sides T_5 and T_6 of T be perpendicular to the z-axis, where the lower subscript represents a closer proximity to the origin.

Let $\mathbf{F} = [F_1, F_2, F_3]$. Then,

$$\iiint\limits_T \operatorname{div} \mathbf{F} dV = \iiint\limits_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz$$

$$= \iiint\limits_T \frac{\partial F_1}{\partial x} dx\, dy\, dz + \iiint\limits_T \frac{\partial F_2}{\partial y} dx\, dy\, dz + \iiint\limits_T \frac{\partial F_3}{\partial z} dx\, dy\, dz$$

$$= \int_{z_1}^{z_2} \int_{y_1}^{y_2} (F_1(x_2, y, z) - F_1(x_1, y, z))\, dy\, dz + \int_{z_1}^{z_2} \int_{x_1}^{x_2} (F_2(x, y_2, z) - F_2(x, y_1, z))\, dx\, dz$$

$$+ \int_{y_1}^{y_2} \int_{x_1}^{x_2} (F_3(x, y, z_2) - F_3(x, y, z_1))\, dx\, dy.$$

$$= \iint\limits_{T_2} F_1\, dy\, dz - \iint\limits_{T_1} F_1\, dy\, dz + \iint\limits_{T_4} F_2\, dx\, dz - \iint\limits_{T_3} F_2\, dx\, dz + \iint\limits_{T_6} F_3\, dx\, dy - \iint\limits_{T_5} F_3\, dx\, dy$$

With this set up $\mathbf{n}$ can be calculated for each side of the surface T, and is as follows:

For $T_1 : \mathbf{n} = -\mathbf{i}$, so $\mathbf{F} \cdot \mathbf{n} = -F_1$. The area is $dA = dydz$.
For $T_2 : \mathbf{n} = \mathbf{i}$, so $\mathbf{F} \cdot \mathbf{n} = F_1$. The area is $dA = dydz$.
For $T_3 : \mathbf{n} = -\mathbf{j}$, so $\mathbf{F} \cdot \mathbf{n} = -F_2$. The area is $dA = dxdz$.
For $T_4 : \mathbf{n} = \mathbf{j}$, so $\mathbf{F} \cdot \mathbf{n} = F_2$. The area is $dA = dxdz$.
For $T_5 : \mathbf{n} = -\mathbf{k}$, so $\mathbf{F} \cdot \mathbf{n} = -F_3$. The area is $dA = dxdy$.
For $T_6 : \mathbf{n} = \mathbf{k}$, so $\mathbf{F} \cdot \mathbf{n} = F_3$. The area is $dA = dxdy$.

Substituting the above into the right-hand side of the last equation,

$$\iint\limits_{T_2} F_1\, dy\, dz - \iint\limits_{T_1} F_1\, dy\, dz + \iint\limits_{T_4} F_2\, dx\, dz - \iint\limits_{T_3} F_2\, dx\, dz + \iint\limits_{T_6} F_3\, dx\, dy - \iint\limits_{T_5} F_3\, dx\, dy$$

$$= \iint\limits_{T_2} \mathbf{F} \cdot \mathbf{n}\, dA + \iint\limits_{T_1} \mathbf{F} \cdot \mathbf{n}\, dA + \iint\limits_{T_4} \mathbf{F} \cdot \mathbf{n}\, dA + \iint\limits_{T_3} \mathbf{F} \cdot \mathbf{n}\, dA + \iint\limits_{T_6} \mathbf{F} \cdot \mathbf{n}\, dA + \iint\limits_{T_5} \mathbf{F} \cdot \mathbf{n}\, dA$$

$$= \sum_{i=1}^{6} \iint\limits_{T_i} \mathbf{F} \cdot \mathbf{n}\, dA$$

Thus,

$$\iiint\limits_T \operatorname{div} \mathbf{F} dV = \iint\limits_{\partial T} \mathbf{F} \cdot \mathbf{n}\, dA.$$

□

2.3 An Example of the Divergence Theorem

Example 1. *Given the vector-valued function* $\mathbf{F} = [x, y, z-1]$ *and the volume of an object defined as* $x^2+y^2+(z-1)^2 = 9$*, and* $1 \le z \le 4$*, show both sides of the Divergence Theorem [3].*

Calculating the divergence of **F**:

$$\nabla \cdot \mathbf{F} = 3$$

Using the left side of the Divergence Theorem

$$\iiint_T \nabla \cdot \mathbf{F}\mathrm{dv} = \iiint_T 3\mathrm{dv}$$

We will convert to polar coordinates using

$$\begin{aligned} x =& r\cos\theta\sin\phi \\ y =& r\sin\theta\cos\phi \\ z =& 1 + r\cos\theta \end{aligned}$$

Taking the Jacobian

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = r^2 \sin\phi$$

$$\iiint_T 3\mathrm{dv} = \iiint_T 3|\mathrm{J}|\mathrm{d}\theta\mathrm{d}\phi\mathrm{dr} = \int_0^3 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 3\mathrm{r}^2 \sin\phi \,\mathrm{d}\theta\mathrm{d}\phi\mathrm{dr} = 54\pi$$

Solving for the right side of the Divergence Theorem

$$\iint_S \mathbf{F}\cdot\mathbf{n}\mathrm{dA} = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\mathrm{dA} + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\mathrm{dA}$$

Solving for the first surface

$$\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\mathrm{dA} = \iint [x, y, z-1]\cdot[0, 0, -1]dA = 0$$

Solving for the second surface and parametrizing the curve

$$\mathbf{r}(u, v) = [3\cos u \cdot \sin v, 3\sin u \cdot \sin v, 1 + 3\cos u]$$

Solving for the normal vector

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = 9[\cos u \cdot \sin^2 v, \sin u \cdot \sin^2 v, \sin v \cdot \cos v]$$

Substitute for $\mathbf{r}(u, v)$ and $\mathbf{n}$, we then solve

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n} \mathrm{dA} = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \mathbf{F}(\mathbf{r}(\mathrm{u}, \mathrm{v})) \cdot \mathbf{n} \, \mathrm{dudv} = 54\pi$$

$$\iint_S \mathbf{F} \cdot \mathbf{n} \mathrm{dA} = 0 + 54\pi = 54\pi$$

3 Green's Theorem

3.1 History of Green's Theorem

Sometime around 1793, George Green was born [9]. He would later go to school during the years 1801 and 1802 [9]. This meant he only received four semesters of formal schooling at Robert Goodacre's school in Nottingham [9]. Not very much is known of Green except that he helped work in his family's business. He must have been working on mathematics throughout his time in the family business, but it is unclear how exactly he came across the advanced material he had learned. In 1823, Green joined a library in Nottingham and had access to more advanced mathematics [9]. In 1828, he published his own work which was left nearly unnoticed [9]. This work contained what is now known as Green's Theorem, but it was not the main idea of the essay and was not yet considered to be the two-dimensional case of the Divergence Theorem [9]. His magnificent work could not be realized by those around him as it was too advanced. He died in 1840 and it wasn't until 1845 that William Thomson republished Green's work and realized the importance of Green's mathematics [9].

Another prominent mathematician of the 19th century was Augustin-Louis Cauchy. He was born in 1789 and met famous mathematicians such as Laplace and Lagrange by 1802 [7]. Lagrange and Laplace were friends with Cauchy's father and took a great interest in Cauchy's education. They insisted he learn classical languages at École Centrale du Panthéon [7]. In 1805, he attended École Polythechnique and had André-Marie Ampère as his tutor [7]. Cauchy wrote his first paper in 1811 and by 1816 had solved a claim by Pierre de Fermat on polygonal numbers [7]. Over the next 30 years, Cauchy produced hundreds of papers. In 1846, he proved Green's Theorem while proving Cauchy's Integral Theorem [4]. Although he provided a proof for this theorem, it was not recognized for many years. Cauchy would still go on to produce a number of other great works and would leave behind a legacy of being one of the greatest mathematicians of all time.

3.2 A Proof of Green's Theorem

Green's Theorem. *Let $\partial R \in \mathbb{R}^2$ be a closed bounded, piecewise smooth, positively oriented simple curve and let R be in the interior of ∂R. Let $F_1(x, y)$ and $F_2(x, y)$ be continuous functions with continuous first partial derivatives everywhere in R. Then*

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_{\partial R} (F_1 dx + F_2 dy).$$

This result is the same as the left side of the Divergence Theorem and therefore they are both equal.

Proof. Proving this theorem for a rectangular area in $\mathbb{R}^2$ will in fact prove the theorem for any arbitrary region, as the nature of the Riemann sums of the double integral ensures this.

Let $R = \{(x, y)|\ x_1 < x < x_2,\ y_1 < y < y_2\}$ with its boundary ∂R orientated counterclockwise. Splitting ∂R into four pieces, one gets ∂R_1 which goes from (x_1, y_1) to (x_2, y_1), ∂R_2 which goes from (x_2, y_1) to (x_2, y_2), ∂R_3 which goes from (x_2, y_2) to (x_1, y_2), and ∂R_4 which goes from (x_1, y_2) to (x_1, y_1). Now

$$\iint_R \frac{\partial F_2}{\partial x} dxdy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} \frac{\partial F_2}{\partial x} dxdy = \int_{y_1}^{y_2} (F_2(x_2, y) - F_2(x_1, y))\, dy = \int_{\partial R_2} F_2(x_2, y)dy + \int_{\partial R_4} F_2(x_1, y)dy.$$

The value of y along ∂R_1 and ∂R_3 is constant, so

$$\int_{\partial R_1} F_2(x, y)dy = \int_{\partial R_3} F_2(x, y)dy = 0,$$

so

$$\int_{\partial R_2} F_2(x_2, y)dy + \int_{\partial R_4} F_2(x_1, y)dy = \int_{\partial R_1} F_2(x, y)dy + \int_{\partial R_2} F_2(x_2, y)dy + \int_{\partial R_3} F_2(x, y)dy + \int_{\partial R_4} F_2(x_1, y)dy = \oint_{\partial R} F_2 dy.$$

In a similar fashion, it can be seen that

$$\iint_R \frac{\partial F_1}{\partial y} dxdy = -\oint_{\partial R} F_1 dx.$$

Thus,

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_{\partial R} (F_1 dx + F_2 dy).$$

□

3.3 An example of Green's Theorem

Example 2. *Given the vector-valued function $\mathbf{F} = [x + y^2, x^2 - y^2]$ and where the curve is a counterclockwise boundary defined as $1 \leq y \leq 2 - x^2$, show both sides of Green's Theorem [3].*

$1 \leq y \leq 2 - x^2$ depicts the region bounded by $y = 1$ and $y = 2 - x^2$.
The points of intersection are $x = \pm 1$.

$$\frac{\partial \mathbf{F_2}}{\partial x} = 2x$$

$$\frac{\partial \mathbf{F_1}}{\partial y} = 2y$$

$$\iint_R \left(\frac{\partial \mathbf{F_2}}{\partial \mathrm{x}} - \frac{\partial \mathbf{F_1}}{\partial \mathrm{y}}\right)\mathrm{dxdy} = \int_{-1}^{1}\int_{1}^{2-x^2} (\mathrm{2x\text{-}2y})\mathrm{dydx} = \frac{-56}{15}$$

Solving for the right side of Green's Theorem using the definition of a vector line integral and parametrizing the curves we have:

$$\begin{aligned} C_1 : \mathbf{r}_1(t) &= [t, 1] \\ \mathbf{r}_1'(t) &= [1, 0] \\ C_2 : \mathbf{r}_2(t) &= [t, 2 - t^2] \\ \mathbf{r}_2'(t) &= [1 - 2t] \\ C &= C_1 + C_2 \end{aligned}$$

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(\mathrm{t})) \cdot \mathbf{r}'(\mathrm{t})\mathrm{dt}$$

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}_1(\mathrm{t})) \cdot \mathbf{r}'_1(\mathrm{t})\mathrm{dt} + \int_b^a \mathbf{F}(\mathbf{r}'_2(\mathrm{t})) \cdot \mathbf{r}'_2(\mathrm{t})\mathrm{dt}$$

$$= \int_{-1}^{1} [\mathrm{t}^2 + 1, \mathrm{t}^2\text{-}1] \cdot [1, 0]\mathrm{dt} + \int_{1}^{-1} [\mathrm{t}^2 + (2\text{-}\mathrm{t}^2)^2, \mathrm{t}^2\text{-}(2\text{-}\mathrm{t}^2)^2] \cdot [1, \text{-}2\mathrm{t}]\mathrm{dt} = \frac{-56}{16}$$

This result is the same as the left side of Green's Theorem and therefore they are both equal.

4 Stokes' Theorem

4.1 History of Stokes' Theorem

George Green not only discovered Green's Theorem, but also stated what would be known as Stokes' Theorem. As mentioned earlier, William Thomson rediscovered the importance of Green's work, and he also found that it contained Stokes' Theorem [9]. Thomson, also known as Lord Kelvin, was born in 1824 and was a child prodigy [13]. At the age of ten he attended Glasgow University and in 1841 attended the University of Cambridge [13]. It is at Cambridge that Kelvin came across Green's work. In 1846, Kelvin became a professor at Cambridge and collaborated with a man named George Gabriel Stokes [13]. Kelvin and Stokes began sharing their thoughts and ideas including the work of Green. Kelvin continued in the direction of physics and engineering, making significant progress in the theories of heat, electricity, and magnetism. The units on the absolute temperature scale are named after Kelvin [13].

Stokes was born on August 13, 1819, in Skreen, Ireland [14]. Stokes' father was a priest, as became all three of his brothers [14]. At the age of 16, he attended Bristol College in England, and then attended Cambridge University [14]. Upon graduating he received a fellowship from Pembroke College [14]. Stokes collaborated with Kelvin and together they became interested in the work of Green. In 1854, Stokes decided to put the theorem as a problem on one of his exams [2]. It is unclear whether the theorem was ever proved by one of his students, but the first known written proof was by Hermann Hankel in 1861 [4]. Stokes fell in love with a woman named Mary Susanna Robinson, and his interest in mathematics steadily decreased [14]. Stokes eventually was drawn back to mathematics and became president of the Royal Society in 1885 [14]. As O'Connor and Robertson stated, "Stokes received the Copley medal from the Royal Society in 1893 and he was given the highest possible honour by his College when he served as Master of Pembroke College" [14]. Stokes died in 1903 in Cambridge, England [14].

Hankel was born February 14, 1839, in Halle, Germany [10]. His father was a professor of physics at the University Leipzig and enrolled Hankel at the Nicolai Gymnasium [10]. In 1857, Hankel was accepted into the University of Leipzig [10]. He studied mathematics under August Möbius and physics under his father [10]. He then studied at the University of Göttingen where he studied under Georg Friedrich Bernhard Riemann [10]. In 1861, Hankel proved Stokes' Theorem "in a treatise on the motion of fluids" [1]. Hankel received his doctorate in 1862 and began teaching at Leipzig in 1863 [10]. He then moved to teach at Tübingen in 1869 [10]. He would continue on to write several papers of great importance, but only few would be noticed because he was consistently making errors in them [10]. He would eventually pass away near Tübingen, Germany in 1873 [10]. Although Hankel did not live as long as many other famous mathematicians, he would still be the first to provide a valid proof of Stokes' Theorem.

4.2 A Proof of Stokes' Theorem

Stokes' Theorem. *Let S be a smooth piecewise oriented surface with its boundary in $\mathbb{R}^3$, and let its boundary, ∂S, be a simple closed piecewise smooth curve. Let $\mathbf{F}$ be a continuous vector-valued function with continuous first partial derivatives in the space containing S. Then*

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} \, dA = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{r} \colon \mathbb{R}^2 \to \mathbb{R}^3$ is a parametrization of S in some region T in the plane.

Proof. Taking the line integral,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r},$$

and breaking it into its components,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \oint_{\partial T} \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial x} dx + \oint_{\partial T} \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial y} dy.$$

Defining $\mathbf{D} = (D_1, D_2) = \left(\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial x}, \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial y}\right)$, then substituting into the line integral

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \oint_{\partial T} D_1 dx + \oint_{\partial T} D_2 dy = \oint_{\partial T} \mathbf{D} \cdot d\mathbf{s}.$$

Now,

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dA = \iint_T (\operatorname{curl} \mathbf{F}) \cdot \left(\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}\right) dxdy = \iint_T \left(\frac{\partial D_2}{\partial x} - \frac{\partial D_1}{\partial y}\right) dxdy.$$

Applying Green's Theorem,

$$\iint_T \left(\frac{\partial D_2}{\partial x} - \frac{\partial D_1}{\partial y}\right) dxdy = \oint_{\partial T} \mathbf{D} \cdot d\mathbf{s},$$

so

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dA = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

□

4.3 An Example of Stokes' Theorem

Example 3. *Given the vector-valued function* $\mathbf{F} = [z^2, x^2, y^2]$ *and a surface defined as* $z^2 = x^2 + y^2$ *where* $y > 0$, *and* $0 \le z \le 2$, *show both sides of Stokes' Theorem [3].*

Taking the cross product

$$\nabla \times \mathbf{F} = [2y, 2z, 2x]$$

Parametrizing the surface

$$\mathbf{r}(u, v) = [v \cos u, v \sin u, v]$$

Calculating the normal vector

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = [-v \cos u, v \sin u, v]$$

Using the left side of Stokes' Theorem

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \mathrm{dA} = \int_0^2 \int_0^\pi [2\mathrm{v}\sin\mathrm{u}, 2\mathrm{v}, 2\mathrm{v}\cos\mathrm{u}] \cdot [\text{-}\mathrm{v}\cos\mathrm{u}, \mathrm{v}\sin\mathrm{u}, \mathrm{v}]\mathrm{dudv} = \frac{\text{-}32}{3}$$

Solving for the right side of Stokes' Theorem using the definition of a vector line integral

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_3} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Parametrizing the three curves

$$\begin{aligned}
C_1 : \mathbf{r}_1(t) &= [t, 0, t] \\
\mathbf{r}_1'(t) &= [1, 0, 1] \\
C_2 : \mathbf{r}_2(t) &= [2\cos t, 2\sin t, 2] \\
\mathbf{r}_2'(t) &= [2\cos t, 2\sin t, 2] \\
C_3 : \mathbf{r}_3(t) &= [t, 0, -t] \\
\mathbf{r}_3'(t) &= [1, 0, -1]
\end{aligned}$$

Substitute and solve

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \frac{8}{3} - 16 + \frac{8}{3} = \frac{-32}{3}$$

This result is the same as the left side of Stokes' Theorem and therefore they are both equal.

5 Applications

There are many far reaching applications that these proofs have contributed to. Many of these applications involve physics and engineering. These equations include: Ampere's Law, Gauss' Law, Gauss' Law for Gravity, Gauss' Law for Magnetism, Heat Flow Equation, and the Maxwell-Faraday Equation of Induction [5]. The equations for these can be seen here.

Ampere's Law.

$$\oint_{\partial S} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I + \mu_0 \epsilon_0 \iint_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{A}$$

Gauss's Law.

$$\oiint_{\partial V} \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\epsilon_0}$$

Gauss's Law for Gravity.

$$\oiint_{\partial V} \mathbf{g} \cdot d\mathbf{A} = -4\pi GM$$

Gauss's Law for Magnetism.

$$\oiint_{\partial V} \mathbf{B} \cdot d\mathbf{A} = 0$$

Heat Flow.

$$\iint_{S} \mathbf{V} \cdot \mathbf{n} d\mathbf{A} = -k \iiint_{T} \nabla^2 \mathbf{U} dxdydz$$

Maxwell-Faraday Equation of Induction.

$$\oint_{\partial S} \mathbf{E} \cdot d\mathbf{l} = -\iint_{S} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$$

6 Epilogue

In the 17th century, Isaac Newton and Gottfried Wilhelm Leibniz helped lead mathematics to the discovery of calculus. Because of their discoveries, mathematics has been changed forever. Also, because of the development of culture at this time, the scientific community became more open to sharing ideas. This increase of interaction between many mathematicians lead to the development of countless discoveries. The combination of the discovery of calculus and increased communication led to a greater number of developments than ever before. It can be seen that these theorems were established because of many intelligent mathematicians. Only the most influential mathematicians relating to these theorems have been named, but there are many more who have not. Lagrange was the first to be on the hunt for a proof of the Divergence Theorem, and although he was unable to prove it, he began a journey that would not be completed for many years. Gauss was the next to take the mantle and made great advances in both mathematics and physics. Ostrogradsky was the first to give a formal proof on the theorem, but it would not have been possible without those who went before him. Green's Theorem became the next big challenge to the area we now know as applied mathematics. Green, whom the theorem is named after, was the first to propose the theorem, but his work would have been forgotten without Thomson. However, it was Cauchy who actually proved the theorem while involved with his own self-titled Cauchy's Integral Theorem. Applied mathematics then took a much more all-purpose approach by discovering the general case which became known as Stokes' Theorem. It was Green who first proposed the theorem, and Thomson who realized the vast importance a theorem of this power could have. Thomson then made this theorem well known through communication with others such as Stokes. Stokes used the theorem as a problem on one of his exams and thus the theorem became known as Stokes' Theorem. Several years later, a gentleman named Hankel proved the theorem, and applied mathematics now contained a great amount of importance and could be used to solve many problems in physics, engineering, and mathematics. These revolutionary theorems are all the outcomes of several different intelligent mathematicians communicating with each other and building on each other's work. None of these theorems were discovered and proven by a single man at a point in time, but rather several men over many years. These theorems have had vast influence in the areas of science and mathematics and so have the great minds that contributed to them. Some of these applications involve electricity, magnetism, gravity, and even heat flow. Because of these theorems, we now have explanations and solutions for problems that were only imagined before. These theorems are very well known, but the history and formation of these theorems are often forgotten. These mathematicians are often known as some of the greatest minds in history, but their impact in the world of applied mathematics is often neglected. These men deserve much more respect than they are normally given. The various contributors that have worked on these theorems have achieved a level of greatness that most will only have the privilege to marvel at. Because of these great developments, our knowledge in mathematics, physics, engineering, and technology will continue to improve.

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