Solutions

# Math 303: Midterm Exam

March 29, 2019

- Write your name and Student ID number in the space provided below and sign.

| Name, Last Name: | |
|---|---|
| ID Number: | |
| Signature: | |

- You have 2.5 hours.
- You must show the details of all your work. Illegible and ambiguous explanations and calculations will lead to deductions from your grade.
- You may use the option of grading your own work. If your estimated grade differs from your actual grade by less than 10 points, you will be given the higher of the two.

| Estimated Grade: | |
|---|---|
| Actual Grade: | |
| Adjusted Grade: | |

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**Problem 1** (10 points) Let **i** and **j** respectively label the unit vectors pointing along the $x$- and $y$-axes in a plane, $f(z) := \sin z$ for all $z \in \mathbb{C}$, and $u(x,y) := \mathrm{Re}(f(x+iy)$. Find the directional derivative of $u$ along the direction given by the vector $\mathbf{n} = \frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})$ at the point $(x_0, y_0) = \cancel{(\ln 2, \pi/6)}\ (\frac{\pi}{6}, \ln 2)$

$$\sin(x+iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{-y}e^{ix} - e^{y}e^{-ix}}{2i}$$

$$= \frac{1}{2i}\left[e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right]$$

$$= \frac{1}{2}(e^{-y}+e^{y})\sin x - \frac{i}{2}(e^{-y}-e^{y})\cos x\,]$$

$$= \cosh y \sin x + \sinh y \cos x$$

$$\Rightarrow u(x,y) = \cosh y \sin x$$

$$(D_{\hat{n}}u)(x,y) = \hat{n}\cdot\vec{\nabla}u(x,y)$$

$$\vec{\nabla}u = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} = \cosh y \cos x\,\hat{i} + \sinh y \sin x\,\hat{j}$$

$$\Rightarrow (D_{\hat{n}}u)(x,y) = \frac{1}{\sqrt{2}}(\cosh y \cos x - \sinh y \sin x)$$

$$(D_{\hat{n}}u)(\tfrac{\pi}{6}, \ln 2) = \frac{1}{\sqrt{2}}\left[\cosh(\ln 2)\frac{\sqrt{3}}{2} - \sinh(\ln 2)\frac{1}{2}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{2}(2+\tfrac{1}{2})(\tfrac{\sqrt{3}}{2}) - \frac{1}{2}(2-\tfrac{1}{2})(\tfrac{1}{2})\right] = \frac{1}{8\sqrt{2}}(5\sqrt{3}-3)$$

## Problem 2:

Let $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ be a solution of the Laplace equation in $\mathbb{R}^2$, i.e., $\forall \vec{x} \in \mathbb{R}^2$, $\nabla^2\varphi(\vec{x}) = 0$. Then $\varphi$ has no maximum or minimum values.

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Proof: First prove that $\varphi$ has no maximum values. Suppose by contradiction that $\varphi$ has a maximum value. Then $\exists \vec{p} \in \mathbb{R}^2$, $\forall \vec{x} \in \mathbb{R}^2$, $\varphi(\vec{p}) \geq \varphi(\vec{x})$ ①

In particular, $\vec{\nabla}\varphi(\vec{p}) = \vec{0}$. ②

Let $f: \mathbb{C} \to \mathbb{C}$ be defined by $\boxed{f(x+iy) := \varphi_y(x,y) + i\varphi_x(x,y)}$

$u := \varphi_y$ & $v := \varphi_x$ $\Rightarrow$ $u_x = \varphi_{yx} = v_y$

& $u_y = \underbrace{\varphi_{yy} = -\varphi_{xx}}_{\text{Because } \nabla^2\varphi = 0} = -v_x$

$\Rightarrow$ $u$ & $v$ satisfy Cauchy-Riemann conditions &

$f(x+iy) = u(x,y) + iv(x,y)$ $\forall (x,y) \in \mathbb{R}^2$ $\hookrightarrow$ $f$ is holomorphic

$\Rightarrow$ its zeros are isolated

$\Rightarrow$ $\vec{p}$ is an isolated zero of $\vec{\nabla}\varphi$ $\Rightarrow$ $\exists \epsilon_1 > 0$ such that $\forall \vec{x} \in N_{\epsilon_1}(\vec{p}) := \{\vec{x}' \in \mathbb{R}^2 \mid |\vec{x}' - \vec{p}| < \epsilon_1\}$, $\left[\begin{array}{c}\vec{x} \neq \vec{p} \\ \Rightarrow \Downarrow \\ \vec{\nabla}\varphi(\vec{x}) \neq \vec{0}\end{array}\right]$

$\Rightarrow$ $N_{\epsilon_1}(\vec{p})$ does not contain any max. point of $\varphi$ except $\vec{p}$.

$\Rightarrow$ $\forall \vec{x} \in N_{\epsilon_1}(\vec{p}) \setminus \{\vec{p}\}$, $\varphi(\vec{x}) < \varphi(\vec{p})$. ③

Now, let $\epsilon_2 > 0$ & $\psi(\vec{x}) := \varphi(\vec{x}) + \epsilon_2(\vec{x} - \vec{p})^2$. Then $\nabla^2\psi = 4\epsilon_2 > 0$ $\Rightarrow$ it is not possible to have $\psi_{xx} \leq 0$ & $\psi_{yy} \leq 0$

$\Rightarrow$ $\psi$ has no local max. points. Moreover, $\psi$ is continuous in $D := \{\vec{x} \in \mathbb{R}^2 \mid |\vec{x} - \vec{p}| \leq \frac{\epsilon_1}{2}\}$ and $D$ is closed and bounded $\Rightarrow$ $\psi$ has a maximum at the boundary of $D$, i.e.,

$C := \partial D = \{\vec{x} \in \mathbb{R}^2 \mid |\vec{x} - \vec{p}| = \frac{\epsilon_1}{2}\}$ $\Rightarrow$ $\exists \vec{q} \in C$, $\forall \vec{x} \in D$,

$\psi(\vec{x}) \leq \psi(\vec{q})$. $\Rightarrow$ $\varphi(\vec{p}) = \psi(\vec{p}) \leq \psi(\vec{q}) = \varphi(\vec{q}) + \frac{\epsilon_2\epsilon_1^2}{4}$

Because $\epsilon_2$ is arbitrary this implies $\varphi(\vec{p}) \leq \varphi(\vec{q})$.

Also ③ holds for $\vec{x} = \vec{q}$ $\Rightarrow$ $\varphi(\vec{q}) < \varphi(\vec{p})$ $\hookrightarrow$ Contradiction.

This proves that $\varphi$ has no maximum value.

To prove that it does not have a minimum value. Let $\tilde{\varphi}(\vec{x}) := -\varphi(\vec{x})$. Clearly $\nabla^2 \tilde{\varphi} = 0 \Rightarrow$ $\tilde{\varphi}$ has no maximum value $\Rightarrow \forall \vec{p} \in \mathbb{R}^2, \exists \vec{x} \in \mathbb{R}^2 \ni$

$$\tilde{\varphi}(\vec{x}) > \tilde{\varphi}(\vec{p}) \Rightarrow -\varphi(\vec{x}) > -\varphi(\vec{p})$$

$$\Rightarrow \varphi(\vec{x}) < \varphi(\vec{p})$$

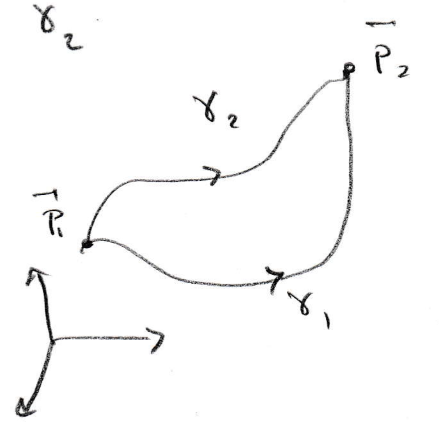
$\Rightarrow \varphi$ has no minimum value. $\boxtimes$

**Problem 3** (15 points) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a function with continuous partial derivatives at every $\mathbf{x} \in \mathbb{R}^3$ and components $F_j$, i.e., $\mathbf{F}(\mathbf{x}) = F_1(\mathbf{x})\mathbf{i} + F_2(\mathbf{x})\mathbf{j} + F_3(\mathbf{x})\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are respectively unit vector pointing along the $x$-, $y$-, and $z$-axes in a Cartesian coordinate system. Suppose that for all $i, j \in \{1,2,3\}$, $\partial_i F_j = \partial_j F_i$, where $\partial_i := \frac{\partial}{\partial x_j}$, $x_1 := x$, $x_2 := y$, and $x_3 := z$. Use Stokes' theorem to show that $\mathbf{F}$ defines a conservative force.

Suppose by contradiction that $\vec{F}$ is not conservative, then there will be different paths with identical end points along which $\vec{F}$ does different amount of work, i.e., $\exists\, \gamma_1, \gamma_2 : [t_1, t_2] \to \mathbb{R}^3$

$\vec{P}_1 := \gamma_1(t_1) = \gamma_2(t_1)$ & $\vec{P}_2 := \gamma_1(t_2) = \gamma_2(t_2)$ such that

$$W_1 := \int_{\gamma_1} \vec{F}\cdot d\vec{r} \neq W_2 := \int_{\gamma_2} \vec{F}\cdot d\vec{r} \quad (1)$$



If $\gamma$ is the closed curve obtained by joins $\gamma_1$ to $\gamma_2$ with $\gamma_2$'s orientation revers

$$(1) \Rightarrow \oint_\gamma \vec{F}\cdot d\vec{r} = \int_{\gamma_1} \vec{F}\cdot d\vec{r} - \int_{\gamma_2} \vec{F}\cdot d\vec{r} \neq 0 \quad (2)$$

Now suppose $S$ is a surface whose boundary is $\gamma$. Then Stokes' theorem states that

$$\oint_\gamma \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma.$$

Because $\partial_i F_j = \partial_j F_i$, $\vec{\nabla}\times\vec{F} = \vec{0}$ $\hookrightarrow$ $\oint_\gamma \vec{F}\cdot d\vec{r} = 0 \quad (3)$

(2) & (3) contradict one another $\Rightarrow$ $\vec{F}$ is conservative.

$\boxtimes$

**Problem 4** (20 points) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, $u(x,y) := \mathrm{Re}[f(x+iy)]$, $v(x,y) := \mathrm{Im}[f(x+iy)]$, and $\mathbf{F}(x,y) = u(x,y)\mathbf{i} + v(x,y)\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are respectively the unit vectors pointing along the $x$- and $y$-axes. Determine the general expression for $f(z)$ such that $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$. You are asked to give an explicit formula for $f(z)$.

$f$ is entire $\Rightarrow$ Cauchy-Riemann conditions hold $\forall (x,y) \in \mathbb{R}^2$

$$u_x = v_y \quad \& \quad u_y = -v_x$$

$$\Downarrow$$

$$0 = \bar{\nabla}\cdot\bar{F} = u_x + v_y = 2u_x \Rightarrow u_x = 0$$

$$\Downarrow$$

$\Rightarrow$ $\boxed{u(x,y) = g(y)}$ for some function $g : \mathbb{R} \to \mathbb{R}$

$$v_y = u_x = 0 \Rightarrow v_y = 0$$

$\Rightarrow$ $\boxed{v(x,y) = h(x)}$ for some " $h : \mathbb{R} \to \mathbb{R}$

Now impose $u_y = -v_x$

$$\Downarrow$$

$$g'(y) = -h'(x)$$

$\Rightarrow$ $g'(y) = c_1$ for some $c_1 \in \mathbb{R}$

$$\Downarrow$$

$\boxed{g(y) = c_1 y + c_2}$ for some $c_2 \in \mathbb{R}$

& $h'(x) = -g'(y) = -c_1$

$$\Downarrow$$

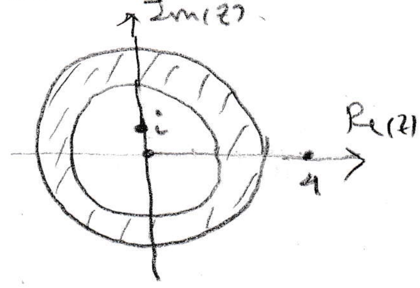
$\boxed{h(x) = -c_1 x + c_3}$ for some $c_3 \in \mathbb{R}$

$\Rightarrow$ $f(z) = f(x+iy) = c_1 y + c_2 + i(-c_1 x + c_3)$

$= -ic_1(x+iy) + (c_2 + ic_3)$ let $z_0 := c_2 + ic_3$

$\Rightarrow$ $\boxed{f(z) = -ic_1 z + z_0}$ for some $c_1 \in \mathbb{R}$ & $z_0 \in \mathbb{C}$

**Problem 5** (25 points) Let $f : \mathbb{C} \to \mathbb{C}$ be the function defined on $\mathbb{C} \setminus \{0, 4, i\}$ according to $f(z) := \dfrac{z^2}{z-i} - \dfrac{1}{z^2(z-4)}$, and $A := \{z \in \mathbb{R} \,|\, 2 \le |z| \le 3\}$. Determine the coefficients of the first two terms in the principal part of the Laurent series expansion of $f(z)$ about $z_0 = 0$ for all $z \in A$. Recall that the principal part of this Laurent series has the form $\sum_{n=1}^{\infty} \dfrac{b_n}{z^n}$. Therefore you are asked to find $b_1$ and $b_2$.




$$\forall z \in A \Rightarrow 2 \le |z| \le 3$$

$$\Rightarrow |i| = 1 < |z| \Rightarrow \left|\frac{i}{z}\right| < 1$$

$$\Rightarrow \frac{z^2}{z-i} = \frac{z}{1-\frac{i}{z}} = z \sum_{m=0}^{\infty} \left(\frac{i}{z}\right)^m$$

Also $|z| < 4 \Rightarrow \left|\frac{z}{4}\right| < 1 \Rightarrow$

$$\Rightarrow \frac{1}{z^2(z-4)} = \frac{1}{(-4z^2)(1-\frac{z}{4})} = -\frac{1}{4z^2} \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n$$

$$\Rightarrow f(z) = z \sum_{m=0}^{\infty} \frac{i^m}{z^m} + \frac{1}{4z^2} \sum_{n=0}^{\infty} \frac{z^n}{4^n}$$

$$= z + i + \sum_{m=2}^{\infty} \frac{i^m}{z^{m-1}} + \frac{1}{4z^2} + \frac{1}{16z} + \sum_{n=2}^{\infty} \frac{z^{n-2}}{4^{n+1}}$$

$$= \underbrace{\left[ i + z + \sum_{n=2}^{\infty} \frac{z^{n-2}}{4^{n+1}} \right]}_{\text{holomorphic part}} + \underbrace{\left[ \frac{1}{4z^2} + \frac{1}{16z} + \frac{-1}{z} + \frac{-i}{z^2} + \sum_{m=4}^{\infty} \frac{i^m}{z^{m-1}} \right]}_{\text{Principal part}}$$

$$\| $$

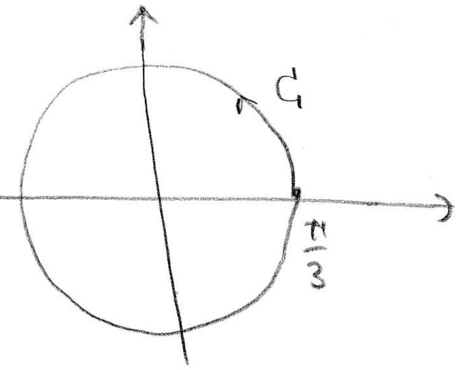
$$\frac{-15}{16z} + \left(-i + \frac{1}{4}\right) \frac{1}{z^2} + \sum_{m=4}^{\infty} \frac{i^m}{z^{m-1}}$$

$$\Rightarrow \boxed{b_1 = -\frac{15}{16} \quad \& \quad b_2 = \frac{1}{4} - i}$$

**Problem 6** (10 points) Use the residue theorem to evaluate $\oint_C \frac{\tan z}{z^2+\sin^2 z}\,dz$, where $C$ is the counterclockwise oriented contour: $\left\{z \in \mathbb{C} \,\middle|\, |z| = \frac{\pi}{3}\right\}$.

**Note:** You may use the fact that $\tan z$ is a holomorphic function everywhere in $\mathbb{C}$ except at the points $z = \frac{(2n+1)\pi}{2}$ where $n$ is an integer, and that $z^2 + \sin^2 z \neq 0$ for $0 < |z| \leq \pi/3$.

$z = 0$ is the only singularity of the integrand inside $C$.




$$\lim_{z\to 0} z\left[\frac{\tan z}{z^2+\sin^2 z}\right] = \lim_{z\to 0}\left(\frac{z^2}{2z^2}\right) = \frac{1}{2} \neq 0$$

$z = 0$ is a simple pole with residue $\frac{1}{2}$

$$\Rightarrow \oint_C \frac{\tan z}{z^2+\sin^2 z}\,dz = 2\pi i\left(\frac{1}{2}\right) = \pi i.$$